Questions:

1. (a) Suppose two sprinters racing each other finish in a tie. Explain, using the Mean Value Theorem, why this means there must have been a moment in the race when the two sprinters were running at exactly the same speed.

> **Solution:** We are told that this has to be true, so let's try to see why. It's first define the position function of the two runners as $A(t)$ and $B(t)$ for the two runners with t being the time since the start of the race. Further, let's define T to be the time at which they finish the race.
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> The two runners run the entire race and hence there is no point in time where they would "jump" any distance so the position functions are continuous (Teleporting in a sprint is considered cheating). Similarly, we can say that their position graph is differentiable on $(0, T)$ since they have a well defined velocity at every point during the race and the velocity is the derivative of the position function. So now that we know $A(t)$ and $B(t)$ are continuous and differentiable, lets hunt for that point.
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> 1. First, we define a new function $D(t) = A(t) - B(t)$. This difference function is a very neat trick we've seen previously. It allows us to simplify two unknown functions into one function which we know something about.
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> 2. For example, we know that at $t = 0$, the two runners are at the same points, so $D(0) = A(0) - B(0) = 0$. Similarly, the two racers finish the race at the same time and so $D(T) = A(T) - B(T) = 0$ as well.
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> 3. Since $D(t)$ a difference of two continuous and differentiable function, $D(t)$ itself is also continuous and differentiable as well. While we're at it, we find that $D'(t) = A'(t) - B'(t)$ (Subtraction rule).
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> 4. Using the previous two points, we can check off the conditions of Rolle's Theorem and hence we get that there must be a point $c \in (0, T)$ such that $D'(c) = 0$.
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> 5. So that means we have:
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> $$D'(c) = 0$$
> $$A'(c) - B'(c) = 0$$
> $$A'(c) = B'(c).$$
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> At time c, we have that the two derivatives are equal. Since the derivative of position is velocity, we can conclude that at point c, the velocity of the two runners are the same. This is the point we are looking for.

(b) Suppose three sprinters racing each other finish in a tie. Can we conclude that there must have been a moment in the race when the three sprinters were running at exactly the same speed? State your answer, and defend it carefully.

> **Solution:** The previous part shows that we are definitely going to find one such point. But that also means in the worst case scenario, we will have only one such point and the velocities are different at every other point. So if we have a third runner with a different velocity at that point, we won't actually be able to find a time where the three runners are at the same speed at the same time. So our intuition tells us that

this problem is probably false. Let's see if we can construct a counter example. To make life easier, I will do the following: Let $A(t), B(t)$ and $C(t)$ be the position of the three runners. I will define the time to be from $t \in [0, 1]$ to indicate the fraction of the time they used to complete the race and I will define the range to be from $[0, 1]$ as well to indicate the fraction of the distance they have covered over the race.

The reduces the problem to finding three functions with domain $[0, 1]$ and range $[0, 1]$ contain no time where their derivatives are simultaneously equal. The simplest function I can think of with that domain and range are

$$A(t) = t \qquad\qquad B(t) = t^2 \qquad\qquad C(t) = t^3$$

with derivatives

$$A(t) = 1 \qquad\qquad B(t) = 2t \qquad\qquad C(t) = 3t^2$$

Comparing $A(t)$ and $B(t)$, we can see that $A'(t) = B'(t)$ at $t = 0.5$. Note that this is the only solution for when the derivatives are equal. But at $t = 0.5$, we see that $C'(t) \neq 1$ and so at $t = 0.5$ the velocities of the three runners are not equal. But this is the only candidate for time since $A'(t) \neq B'(t)$ for every other t. We can then conclude that it is not always the case that there will be a point in time when the three runners are going at the same velocity.

2. In the 2006 paper "Use of the gamma distribution to represent monthly rainfall in Africa for drought monitoring applications", published in the *International Journal of Climatology*, the authors refer to gamma probability distribution functions of the form

$$f(x) = Mx^a e^{-kx},$$

where M, a and k are positive constants, and $f(x)$ is the probability that a particular location will have a monthly rainfall accumulation x. (This is a simplified form of the function labelled (2) in the paper.)

(a) $f(x)$ (as given above) has exactly one critical point. Find that point.

Solution: To find the critical point, we need to find when the derivative is either zero or undefined. So that means we need to find the derivative.

$$\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}[Mx^a e^{-kx}] \\
&= M\frac{\mathrm{d}}{\mathrm{d}x}\left(x^a e^{-kx}\right) \\
&= M\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(x^a\right)e^{-kx} + x^a \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-kx}\right)\right) \\
&= M\left(ax^{a-1}e^{-kx} + x^a\left(-ke^{-kx}\right)\right) \\
&= M\left(ax^{a-1}e^{-kx} - kx^a e^{-kx}\right) \\
&= Mx^{a-1}e^{-kx}\left(a - kx\right)
\end{aligned}$$

Equipped with the derivative, we can see when $f'(x) = 0$. Since this is a multiplication of a bunch of stuff, for it to be zero or undefined, at least one of the bits have to zero or undefined.

- The initial M is a positive constant and so is neither zero nor undefined.

- The exponential function is always positive and defined, so it does not contribute any critical points.

- The polynomial term x^{a-1} will give a critical point at $x = 0$ most of the time since it will give 0 when $a > 1$ and be undefined when $0 < a < 1$. The only time it does not contribute is when $a = 1$.

- The linear term $(a - kx)$ is defined for all a, k and will be zero when $x = \frac{a}{k}$ since we know $k > 0$.

So that means the critical points are $x = 0, \frac{a}{k}$.

(b) Suppose the critical point in part (a) is denoted c. Is the graph of $f(x)$ increasing or decreasing on the interval (c, ∞)? Justify your answer.

Solution: Since the critical point $x = 0$ might not always exist, the point we are interested in is $c = \frac{a}{k}$. To see if $f(x)$ is increasing or decreasing, we need to check the sign of $f'(x)$ on the interval (c, ∞). To pick a number larger than $\frac{a}{k}$ we can try something like $\frac{a+1}{k}$. Other numbers will be too, but it has to be true for all values of a and k. Using this:

$$f'(x) = Mx^{a-1}e^{-kx}(a - kx)$$

$$f'\left(\frac{a+1}{k}\right) = M\left(\frac{a+1}{k}\right)^{a-1}e^{-k\left(\frac{a+1}{k}\right)}\left(a - k\left(\frac{a+1}{k}\right)\right)$$

$$= M\left(\frac{a+1}{k}\right)^{a-1}e^{-(a+1)}(a - (a+1))$$

$$= M\left(\frac{a+1}{k}\right)^{a-1}e^{-(a+1)}(-1)$$

We can then extract the sign of the above by comparing the sign of each term.

- The constant $M > 0$ and is positive

- The polynomial term is a positive number raise to some power and hence is also positive

- The exponential term is positive

- The last term negative (since its exactly -1).

That means the multiplication of that mess will give a negative number which means $f'(x) < 0$ and hence $f(x)$ must be decreasing on the interval (c, ∞).

Additional Stuff: This is not necessary, but we can show that c is a local max of $f(x)$ by showing that it is increasing on the interval $(0, c)$. This is a little tricky though since the number $x = \frac{a-1}{k}$, while less than $\frac{a}{k}$, might be negative if $a < 1$. To correct for this, we define $0 < q < a$ and consider the number $x = \frac{a-q}{k}$, which will

definitely be in the interval $(0, c)$. (We can use $x = \frac{a+q}{k}$ as well as a point in (c, ∞):

$$f'\left(\frac{a-q}{k}\right) = M\left(\frac{a-q}{k}\right)^{a-1} e^{-(a-q)}(q)$$

$$f'\left(\frac{a+q}{k}\right) = M\left(\frac{a+q}{k}\right)^{a-1} e^{-(a+q)}(-q)$$

which is positive in the first case, and negative in the second case. So $f(x)$ changes from positive to negative at $x = \frac{a}{k}$ which means it's a local max.

(c) Why does your answer in part (b) make sense in terms of the physical phenomenon being described? In other words, why does it make sense for the graph of the probability that a particular location will have a monthly rainfall accumulation x to be decreasing (or increasing) on (c, ∞)? Justify your answer in a few sentences.

Solution: Recall that the function $f(x)$ is the probability of obtaining x units. A decreasing function means that the probability of obtaining x units of rain will decrease as x increases. In other words, there is a smaller probability of obtaining a larger rainfall. In the real of the real world, we expect an area to have some optimal (or expected) amount of rain and the chance of obtaining a rainfall that is significantly larger than the optimal will be smaller than that of obtaining the optimal amount. Indeed, the real world expectations agree with the predictions of the model.

(d) [Bonus] What do you think the area under the graph of the function $f(x)$ (and above the x-axis) from $x = 5$ cm and $x = 50$ cm represents?

Solution: Let's think about what this means, by thinking about the height of the function as little poles/columns/bars above the x value. So the bar above $x = 5$ would be the probability of obtaining a rainfall of 5. Then repeat for $x = 6, x = 7$ etc. The combined height of all the bars will be the total probability of obtaining rainfall that is either $x = 5, 6, 7, \ldots$. Now repeat this for smaller bars to include numbers like $5.1, 5.2, 5.3, \ldots$ and their combined height will be the probability of obtaining $x = 5, 5.1, 5.2 \ldots$. We can repeat this process for smaller and smaller division of the numbers until we get all the numbers. Then the combined height (which is now the area of the graph) is now the probability that the rainfall is greater than 5. So that means the area between $x = 5$cm and $x = 50$cm is the probability the rainfall obtained in the area is between $x = 5$ and $x = 50$.

3. (a) Suppose two functions $f(x)$ and $g(x)$ have the same derivative. Show that f and g are *parallel* in the sense that $h(x) = f(x) - g(x)$ is constant.

Solution: Consider $h(x) = f(x) - g(x)$. We wish to show that h is constant. Recall that a function with zero derivative everywhere is constant. So we wish to show $h'(x) = 0$ for all x. Indeed, $h'(x) = f'(x) - g'(x) = 0$ since f and g have identical derivatives. Hence, h is constant and so functions f and g differ only by a constant.

(b) Describe all functions with derivative e^{2x}.

Solution: Before we can find all functions with the specified derivative we should start by finding just one. Armed with our knowledge of exponential functions and chain rule with a little reverse engineering and trial and error we see that $f(x) = \frac{1}{2}e^{2x}$ has the required derivative. Observe, $f'(x) = \frac{1}{2}e^{2x} \cdot 2 = e^{2x}$. Now from part (a) we know that any two functions with the same derivative differ by a constant. That is to say, not only is $f_c(x) = \frac{1}{2}e^{2x} + c$ for any $c \in \mathbb{R}$ a solution to our problem but that all such solutions can be written in this way. Hence, we have found all functions with the specified derivative.

(c) Find the function $f(x)$ such that $f'(x) = e^{2x}$ and $f(0) = 2$.

Solution: In part (b) we found all functions with derivative e^{2x}. The function we seek must be one of these. We therefore hunt for the c value that will give the desired $f(0) = 2$. If we insist that $2 = f(0) = \frac{1}{2}e^{2 \cdot 0} + c$ we can solve for the required c. Observe,

$$2 = \frac{1}{2}e^0 + c$$
$$2 = \frac{1}{2} + c$$

and so $c = \frac{3}{2}$. And so $f(x) = \frac{1}{2}e^{2x} + \frac{3}{2}$ has the specified properties.