ASSIGNMENT 2·5: Section 002

There are two parts to this assignment. The first part is on WeBWorK — the link is available on the course webpage. The second part consists of the questions on this page. You are expected to provide full solutions with complete arguments and justifications. You will be graded on the correctness, clarity and elegance of your solutions. Your answers must be typed or very neatly written. They must be stapled, with your name and student number at the top of each page.

1. Given the function $y = f(x)$, do the following:
   i. Find the domain of f.
   ii. Find the x-intercepts and y-intercept of the graph of f.
   iii. Find where f is positive or negative.
   iv. Find the intervals where f is increasing ($\uparrow$), and the intervals where f is decreasing ($\downarrow$).
   v. Find all local maxima and all local minima of the function f. State how you know this, i.e. which test did you use?
   vi. Find all intervals where the graph of f is concave up ($\cup$) and all intervals where the graph of f is concave down ($\cap$).
   vii. Find all inflection points of the graph of f.
   viii. Plot each "important point" accurately and then connect the dots by using the information that you have gathered.

   Solution

   For

   $$y = \frac{\cos x}{2 + \sin x}$$

   i) The domain is all real numbers since $2 + \sin x \neq 0$ as $\sin x$ only takes values between -1 and 1.
   ii) The y-intercept is

   $$y = \frac{\cos(0)}{2 + \sin(0)} = \frac{1}{2}$$

   The x-intercepts occur when $\cos x = 0$ so at $x = \frac{\pi}{2} + k\pi$ where k is an integer.
   iii) Since the denominator is always positive the sign of our function will be determined by the numerator. The numerator $\cos x$ is positive on $(-\pi + 2k\pi, \pi + 2k\pi)$ and negative on $(\pi + 2k\pi, 3\pi/2 + 2k\pi)$ where k is an integer.
   iv) Let us inspect the derivative

   $$\begin{aligned} f'(x) &= \frac{-\sin x(2 + \sin x) - \cos x \cos x}{(2 + \sin x)^2} \\ &= \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2 + \sin x)^2} \\ &= \frac{-2\sin x - 1}{(2 + \sin x)^2} \end{aligned}$$

   The derivative always exists, so the critical points will occur when $f'(x) = 0$ which will be the case when the numerator is zero. So

   $$\begin{aligned} 0 &= -2\sin x - 1 \\ \sin x &= -\frac{1}{2} \end{aligned}$$

On the interval $[0, 2\pi]$ the solutions are $x = 7\pi/6$ and $x = 11\pi/6$. The 2π periodicity of $\sin x$ gives us the following solutions on the real line: $x = 7\pi/6 + 2k\pi$ and $x = 11\pi/6 + 2k\pi$ where k is an integer. Since our function is periodic is it sufficient to consider its behaviour on the interval $[0, 2\pi]$. To determine the intervals of increase and decrease we are interested in the sign of the derivative. Since again the denominator is positive always we need only consider the sign on the numerator: $-2\sin x - 1$. We can either draw $-2\sin x - 1$ by transforming the sin function or else consider the following table.

	$[0, 7\pi/6)$	$(7\pi/6, 11\pi/6)$	$(11\pi/6, 2\pi]$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$
$f'(x)$	< 0	> 0	< 0

The above gives our intervals of increase and decrease given that we can extend these results in a periodic way, repeating ever 2π.
v) From the previous part we see that $x = 7\pi/6 + 2k\pi$ is a local minimum and that $x = 11\pi/6 + 2k\pi$ is a local maximum.
vi) For concavity we investigate the second derivative.

$$\begin{aligned} f''(x) &= -\frac{d}{dx}\frac{2\sin x + 1}{(2+\sin x)^2} \\ &= -\frac{2\cos x(2+\sin x)^2 - (2\sin x + 1)\cdot 2(2+\sin x)\cos x}{(2+\sin x)^4} \\ &= -\frac{2\cos x(2+\sin x)\,[2+\sin x - 2\sin x - 1]}{(2+\sin x)^4} \\ &= -\frac{2\cos x\,[1-\sin x]}{(2+\sin x)^3} \\ &= \frac{2\cos x(-1+\sin x)}{(2+\sin x)^3} \end{aligned}$$

To find the possible inflection points we find where $f''(x) = 0$. Note that the denominator, again, is always positive so we need only consider the numerator. We aim to solve

$$\cos x(\sin x - 1) = 0$$

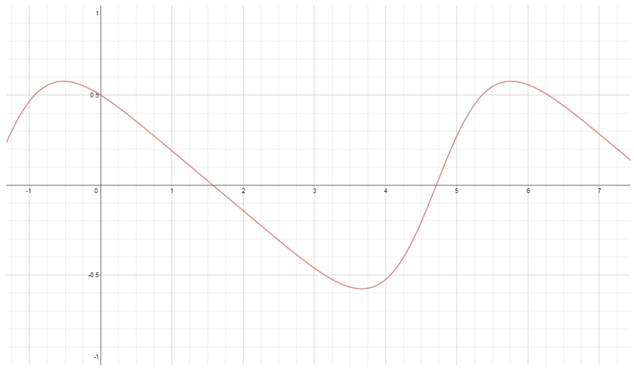
The above equation will be satisfied when either $\cos x = 0$ or $\sin x = 1$. For the second equality $\sin x = 1$ when $x = \pi/2 + 2k\pi$ where k is an integer. For the first $\cos x = 0$ when $x = \pi/2 + k\pi$ where k is an integer. Note that when the second equation is satisfied the first is also satisfied. We consider again the interval $[0, 2\pi]$ and construct the following table. Note that $\sin x - 1$ is always negative or zero and never positive.

	$[0, \pi/2)$	$(\pi/2, 3\pi/2)$	$(3\pi/2, 2\pi]$
$f(x)$	*C.U.*	*C.D.*	*C.U.*
$f''(x)$	> 0	< 0	> 0

The above table gives our intervals of concavity.
vii) Based on the above table we have infection points at $x = \pi/2 + k\pi$ where k is an integer.
viii) See figure.

Now on to

$$y = \frac{x^2}{x^2+3}$$



i) The domain is all real numbers since the denominator x^2+3 is always positive and so never zero.
ii) The y-intercept is

$$y = \frac{0^2}{0^2+3} = 0$$

and the x-intercept occurs when the numerator equals zero so when $x^2 = 0$ which gives $x = 0$.
iii) Both the numerator and denominator are positive except when the function is zero at $x = 0$. The function is never negative.
iv) The derivative

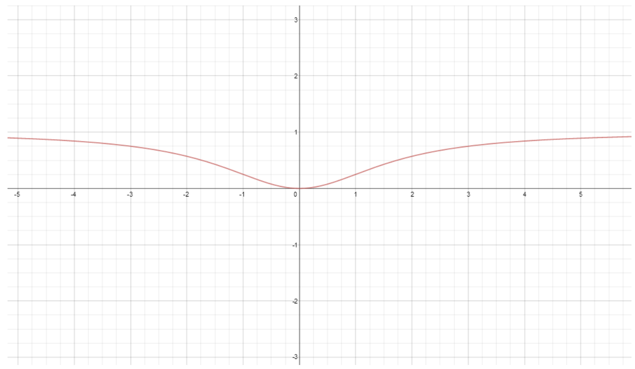
$$\begin{aligned} f'(x) &= \frac{2x(x^2+3) - x^2(2x)}{(x^2+3)^2} \\ &= \frac{2x^3+6x-2x^3}{(x^2+3)^2} \\ &= \frac{6x}{(x^2+3)^2} \end{aligned}$$

The denominator is always positive and the numerator is zero when $x = 0$, our critical point. For intervals of increase and decrease consider the following table

	$(-\infty, 0)$	$(0, \infty)$
$f(x)$	$\searrow$	$\nearrow$
$f'(x)$	< 0	> 0

v) The above table gives $x = 0$ a local minimum.
vi) We compute the second derivative

$$\begin{aligned} f''(x) &= \frac{6(x^2+3)^2 - 6x \cdot 2(x^2+3) \cdot 2x}{(x^2+3)^4} \\ &= \frac{6(x^2+3)^2 - 6 \cdot 4x^2(x^2+3)}{(x^2+3)^4} \\ &= \frac{6(x^2+3)\left[x^2+3-4x^2\right]}{(x^2+3)^4} \\ &= \frac{6(3-3x^2)}{(x^2+3)^3} \\ &= \frac{18(1-x^2)}{(x^2+3)^3} \end{aligned}$$



Again, the denominator is always positive. Our possible inflection points occur when $1 - x^2 = 0$ so $x = \pm 1$. To determine the intervals of concavity consider the following table

	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$f(x)$	$C.D.$	$C.U.$	$C.D.$
$f''(x)$	< 0	> 0	< 0

vii) The above table gives inflection points at $x = -1$ and $x = 1$. viii) See graph.