

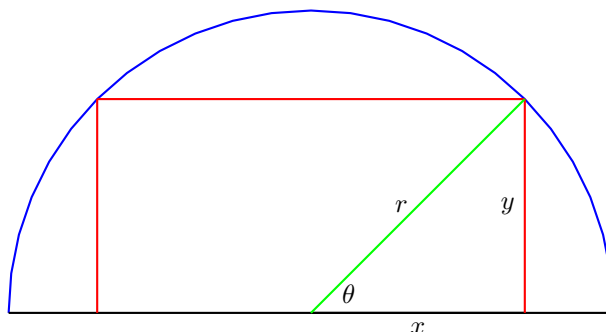
## ASSIGNMENT 2.8 Solutions Section 002

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There are two parts to this assignment. The first part is on WeBWorK — the link is available on the course webpage. The second part consists of the questions on this page. You are expected to provide full solutions with complete arguments and justifications. You will be graded on the correctness, clarity and elegance of your solutions. Your answers must be typed or very neatly written. They must be stapled, with your name and student number at the top of each page.

**Question 1** Find the area of the largest rectangle that can be inscribed in a semicircle of radius  $r$ .

**Solution** Place a rectangle inside a semicircle as shown below. Let  $x$  and  $y$  be as in the figure.



In this way the area of the rectangle is given by

$$A = 2xy.$$

By the way we have arranged the rectangle the point  $(x, y)$  must lie on the circle, which we know to have equation  $r^2 = x^2 + y^2$ . We think of the radius  $r$  to be given and solve for  $y$ . In this way we can eliminate  $y$  from the area equation in favour of  $x$ . So

$$y = \sqrt{r^2 - x^2}$$

and hence

$$A(x) = 2x\sqrt{r^2 - x^2} = 2x(r^2 - x^2)^{1/2}$$

is our area equation. Note that  $x \in [0, r]$  is the natural domain. We will now apply the closed interval method. To find critical points we compute the derivative

$$\begin{aligned} A'(x) &= 2(r^2 - x^2)^{1/2} + x(r^2 - x^2)^{-1/2}(-2x) \\ &= 2(r^2 - x^2)^{1/2} - 2x^2(r^2 - x^2)^{-1/2} \\ &= 2(r^2 - x^2)^{-1/2} (r^2 - x^2 - x^2) \\ &= \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} \end{aligned}$$

The derivative does not exist when  $x = \pm r$ . However, one of these is outside our domain and the other is an endpoint so we are already aware of it. The derivative is equal to zero when  $r^2 = 2x^2$  or rather  $x = r/\sqrt{2}$  where we have taken only the positive root since the negative root is outside our domain. We now evaluate  $A(x)$  at the endpoints and critical points. Observe that  $A(0) = 0$  and  $A(r) = 0$  and

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = \sqrt{2}r\sqrt{\frac{r^2}{2}} = r^2 > 0.$$

Hence, the maximum area of the rectangle is  $r^2$ .

**Alternate Solution** We can solve this problem a little faster if we write the area in terms of the angle  $\theta$  as shown in the figure. In this way  $x = r \cos \theta$  and  $y = r \sin \theta$  so

$$A(\theta) = 2r^2 \cos \theta \sin \theta = r^2 \sin 2\theta$$

where we have used a trig identity. Note that  $\theta \in [0, \pi/2]$ . Using our knowledge of the function  $\sin \theta$  and transformations we see (without the use of calculus) that the max of  $\sin 2\theta$  is 1 which occurs at  $\theta = \pi/4$ . The maximum of the function  $A(\theta)$  is therefore  $r^2 \sin \pi/2 = r^2$ .

**Question 2** Your love has been lost at sea for months. Each day you stand on a cliff, at point  $(0, 0)$ , looking out to sea wondering if they will return. One day you see a boat due north at point  $(0, 2)$ . You notice the boat is traveling down the river along the curve  $y = \sqrt{x+4}$  towards the harbour, which is west of you at point  $(-4, 0)$ . When the boat reaches the point where it is closest to your position you recognize it as your love's boat. Find the coordinates of this point.

**Solution**

The distance between  $(0, 0)$  and a the point  $(x, y)$  on the curve  $y = \sqrt{x+4}$  is

$$\begin{aligned} d &= \sqrt{(x-0)^2 + (y-0)^2} \\ &= \sqrt{x^2 + y^2}. \end{aligned}$$

Since the point  $(x, y)$  is on the curve we can eliminate  $y$  in favour of  $x$  by using the equation of the curve. So

$$\begin{aligned} d &= \sqrt{x^2 + (\sqrt{x+4})^2} \\ &= \sqrt{x^2 + x + 4}. \end{aligned}$$

Rather than minimize  $d(x)$  we instead minimize  $f(x) = d^2(x)$ . Since  $d$  is a positive function it will have the same minimum as  $f$ . We hereafter consider

$$f(x) = x^2 + x + 4$$

with  $x \in [-4, 0]$ . We aim to use the closed interval test and so evaluate  $f$  at the critical points and the endpoints. We compute

$$f'(x) = 2x + 1.$$

So  $f'(x) = 0$  when  $2x + 1 = 0$  or rather  $x = -1/2$ . Note that  $f'(x)$  exists everywhere. We now evaluate

$$\begin{aligned} f(-4) &= (-4)^2 - 4 + 4 = 16 \\ f(-1/2) &= (-1/2)^2 - 1/2 + 4 = 1/4 - 2/4 + 16/4 = 15/4 < 16/4 = 4 \\ f(0) &= 4 \end{aligned}$$

So the minimum of  $f$ , and hence  $d$ , occurs at  $x = -1/2$ . The corresponding  $y$  coordinate is

$$y = \sqrt{-1/2 + 4} = \sqrt{7/2}.$$

The coordinates of the closest point are therefore  $(-1/2, \sqrt{7/2})$ .

**Question 3** You are selling subscriptions to an online magazine. Based on some market research you determine that if you set the price of the subscription to be  $p$  in dollars then  $q(p)$  is the quantity of subscriptions you are able to sell (in thousands) where

$$q(p) = 6 - 2\sqrt{p}.$$

The revenue you will generate is give by

$$r(p) = \text{price} \cdot \text{quantity} = p \cdot q(p).$$

What price should you set in order to maximize revenue?

**Solution** Substituting  $q(p)$  to  $r(p) = p \cdot q(p)$  gives

$$r(p) = p(6 - 2\sqrt{p})$$

Note that we require  $p \in [0, 9]$  since price must be positive and if  $p > 9$  then quantity is negative (we think about this as: the price is to high to sell any product). We now apply the closed interval method to find the minimum of  $r$ . First write

$$r(p) = 6p - 2p^{3/2}.$$

Now compute the derivative

$$r'(p) = 6 - 3p^{1/2}.$$

We have  $r'(p) = 0$  when  $6 - 3\sqrt{p} = 0$  so  $p = 4$  is our critical point. Next, evaluate  $r$  at the critical point and endpoints

$$r(0) = 0$$

$$r(4) = 4(6 - 2\sqrt{4}) = 4(6 - 4) = 4(2) = 8 > 0$$

$$r(9) = 0$$

So, a price of \$4 per subscription will maximize revenue.