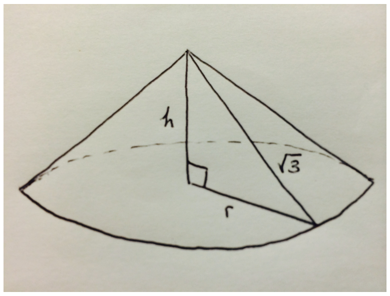
ASSIGNMENT $2 \cdot 9$ Solutions

There are two parts to this assignment. The first part is on WeBWorK — the link is available on the course webpage. The second part consists of the questions on this page. You are expected to provide full solutions with complete arguments and justifications. You will be graded on the correctness, clarity and elegance of your solutions. Your answers must be typed or very neatly written. They must be stapled, with your name and student number at the top of each page.

1. A right triangle of hypotenuse $\sqrt{3}$ is revolved about one of its legs to create a right circular cone. Find the radius, height, and volume of the cone of greatest volume.



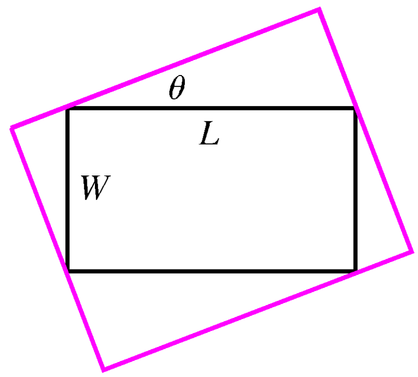
SOLUTION:

Our variables are the height h and base r of the triangle. r is also is the radius of the top of the cone. We have a constant of $\sqrt{3}$ for the hypotenuse of the triangle/length of the cone.
Our constraint equation comes from the Pythagorean Theorem and the cross section triangle, so that $r^2 + h^2 = (\sqrt{3})^2$ or $r^2 + h^2 = 3$. Also, we know that $0 \leq r, h \leq \sqrt{3}$.
We are trying to *maximize* volume, $V = \frac{1}{3}\pi r^2 h$.
Solving for r^2 gives $r^2 = 3 - h^2$, so $V = \frac{1}{3}\pi(3 - h^2)h$ or $V = \frac{1}{3}\pi(3h - h^3)$.
Taking the derivative of V with respect to h yields $V'(h) = \frac{1}{3}\pi(3 - 3h^2) = \pi(1 - h^2)$.
Now, the critical points are $h = \pm 1$, and since the heigh cannot be negative, we only consider $h = 1$.
Using the closed interval method, we'll check $V(0), V(\sqrt{3}), V(1)$. If $h = 0$, then $r = \sqrt{3}$ and $V = 0$. If $h = \sqrt{3}$, then $r = 0$ and $V = 0$. If $h = 1$, the $r^2 = 3 - h^2 = 3 - 1 = 2$, so $r = \sqrt{2}$, and the volume (which is the max) is $V = \frac{2\pi}{3}$.
Alternatively, we could solve for h, but this is trickier. Using our constraint $h = \sqrt{3 - r^2}$, we have that $V = \frac{1}{3}\pi r^2 \sqrt{3 - r^2}$.
So,

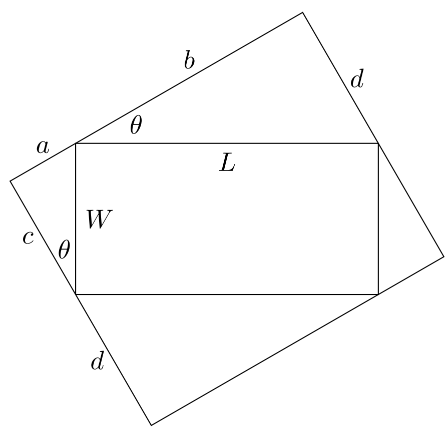
$$\begin{aligned}
V'(r) &= \frac{2}{3}\pi r\sqrt{3 - r^2} + \frac{1}{3}\pi r^2 \left(\frac{-r}{\sqrt{3 - r^2}}\right) \\
&= \frac{2}{3}\pi r \frac{3 - r^2}{\sqrt{3 - r^2}} + \frac{1}{3}\pi r^2 \left(\frac{-r}{\sqrt{3 - r^2}}\right) \\
&= \frac{\pi}{3} \frac{6r - 2r^3 - r^3}{\sqrt{3 - r^2}} \\
&= \frac{r(2 - r^2)}{\sqrt{3 - r^2}}
\end{aligned}$$

So, the critical points are $r = 0, \pm\sqrt{2}, \pm\sqrt{3}$. The only reasonable one is $r = \sqrt{2}$. Using the closed interval method, we'll check $V(0), V(\sqrt{3}), V(1)$. If $h = 0$, then $r = \sqrt{3}$ and $V = 0$. If $h = \sqrt{3}$, then $r = 0$ and $V = 0$. If $r = \sqrt{2}$, so $h = \sqrt{3-2} = 1$ and $V = \frac{2}{3}\pi$.

2. Find the maximum area of a rectangle that can be circumscribed about a given rectangle of length L and width W. *Hint*: One way, but not the only way, to do this is to consider the area of the large rectangle as a function of the angle θ shown below.



SOLUTION: Consider the following rectangle surrounding our given rectangle.



The area of the large rectangle is given by

$$V = (a+b)(c+d).$$

We wish to write V as a function of the angle θ. Using trigonometry we write the labeled side lengths as functions of θ

$$\begin{aligned} a &= W\sin\theta \\ c &= W\cos\theta \\ b &= L\cos\theta \\ d &= L\sin\theta. \end{aligned}$$

Putting everything together gives

$$\begin{aligned} V(\theta) &= (W\sin\theta + L\cos\theta)(W\cos\theta + L\sin\theta) \\ V(\theta) &= W^2\sin\theta\cos\theta + WL\sin^2\theta + WL\cos^2\theta + L^2\sin\theta\cos\theta \\ V(\theta) &= (W^2+L^2)\sin\theta\cos\theta + WL(\sin^2\theta + \cos^2\theta) \\ V(\theta) &= (W^2+L^2)\sin\theta\cos\theta + WL \end{aligned}$$

where our domain is $\theta \in [0, \pi/2]$. We aim to use the closed interval method. We find critical points

$$V'(\theta) = (W^2 + L^2)(\sin^2\theta - \cos^2\theta)$$

We will have $V'(\theta) = 0$ when

$$\sin^2\theta = \cos^2\theta$$

or rather

$$\sin\theta = \pm\cos\theta.$$

Actually, we select only the positive solution since θ in our domain will produce only positive values of $\sin\theta$ and $\cos\theta$. So

$$\sin\theta = \cos\theta.$$

The only θ in our domain satisfying the above is $\pi/4$. With our lone critical point in hand we now check the values of V at $\pi/4$ and at the endpoints.

$$\begin{aligned} V(0) &= WL \\ V(\pi/4) &= (W^2 + L^2)/2 + WL > WL \\ V(\pi/2) &= WL \end{aligned}$$

Hence $V(\pi/4)$ is the largest area of the big rectangle. Alternatively had we noticed the trig identity $2\sin\theta\cos\theta = \sin 2\theta$ we could have rewritten V as

$$V(\theta) = \frac{W^2 + L^2}{2}\sin 2\theta + WL$$

and observed that $\sin(2\theta)$ achieves its max of 1 at $\pi/4$ via a transformation of the sin function.

3. The sound level measured in watts per square meter, varies in direct proportion to the power of the source and inversly as the square of the distance from the source. So the sound level S is given by

$$S = kPd^{-2}$$

where P is the source power, d is the distance from the source, and k is a constant greater than zero.
Two people in my neighborhood decided to throw parties on the same day, 200 meters apart, and are both blasting music on their stereos. The first party has 45 times as much power as the second. If I stand on the line segment between the two parties, where should I go if I want my surroundings to be as quiet as possible? What if I can stand outside the line segment connecting the two parties?
Solution: Let P be the source power of the second party's stereo and let d be the distance between me and the first party. Since the power of the first party's stereo is $45P$, the sound level is:

$$L(d) = 45Pkd^{-2} + kP(200 - d)^{-2}, 0 < d < 200$$

Our domain is $(0, 200)$ since we are standing on the line segment between the two parties. Now, taking the derivative, we obtain

$$L'(d) = 2kP\left(\frac{1}{(200 - d)^3} - \frac{45}{d^3}\right)$$

and our only critical point occurs when $d = \frac{200(45)^{1/3}}{1+(45)^{1/3}}$. Since we are on the open interval $(0, 200)$, we use the first derivative test to find what kind of extreme value $d = \frac{200(45)^{1/3}}{1+(45)^{1/3}}$ is.

$$\text{To the left of } d = \frac{200(45)^{1/3}}{1+(45)^{1/3}} \text{ in } (0,200),\ L(1) < 0.$$
$$\text{To the right of } d = \frac{200(45)^{1/3}}{1+(45)^{1/3}} \text{ in } (0,200),\ L(199) > 0$$

Thus, by the variant of the first derivative test, $d = \frac{200(45)^{1/3}}{1+(45)^{1/3}} \approx 156.1$ is the *global* minimum for $L(d)$.