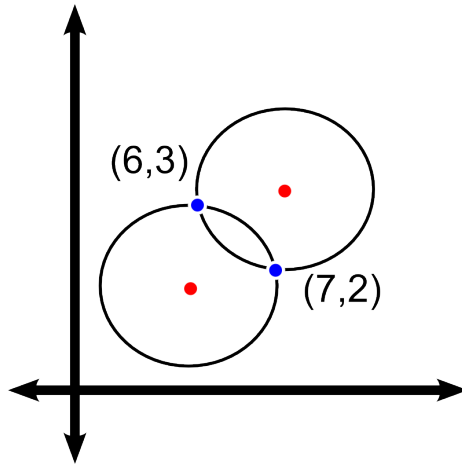


Questions:

1. There are exactly two circles of radius $\sqrt{5}$ through the points $(6, 3)$ and $(7, 2)$. Find the equations of both circles.

Solution: We can approach this from a purely algebraic perspective. But it might help if to draw a picture to highlight the fact that there are indeed two circles.



Recall that the equation of a circle centered at a point (x_0, y_0) with radius r is given by:

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

The question asks us to find two possible values of (x_0, y_0) . Since we are given that $r = \sqrt{5}$, and two points on the circle, we can simply substitute these values into the equation of a circle.

$$(6 - x_0)^2 + (3 - y_0)^2 = (\sqrt{5})^2.$$

$$(7 - x_0)^2 + (2 - y_0)^2 = (\sqrt{5})^2.$$

By expanding out the equations, we get that:

$$(36 - 12x_0 + x_0^2) + (9 - 6y_0 + y_0^2) = 5.$$

$$(49 - 14x_0 + x_0^2) + (4 - 4y_0 + y_0^2) = 5.$$

By taking the difference between the two equations, we get that:

$$(-13 + 2x_0) + (5 - 2y_0) = 0.$$

which we can rearrange to give:

$$2x_0 - 2y_0 - 8 = 0$$

$$2x_0 - 8 = 2y_0$$

$$x_0 - 4 = y_0$$

We now have an relation between y_0 and x_0 which we can use substitute into the original equation. I chose the first one:

$$\begin{aligned}
 (6 - x_0)^2 + (3 - y_0)^2 &= (\sqrt{5})^2 \\
 (6 - x_0)^2 + (3 - (x_0 - 4))^2 &= 5 && \text{Substitution} \\
 (6 - x_0)^2 + (7 - x_0)^2 &= 5 && \text{Simplify} \\
 (36 - 12x_0 + x_0^2) + (49 - 14x_0 + x_0^2) &= 5 && \text{Expanding} \\
 85 - 26x_0 + 2x_0^2 &= 5 && \text{Simplify} \\
 2x_0^2 - 26x_0 + 80 &= 0 \\
 x_0^2 - 13x_0 + 40 &= 0.
 \end{aligned}$$

We can then use our trusty quadratic formula with $a = 1$, $b = -13$ and $c = 40$:

$$\begin{aligned}
 x_0 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 &= \frac{-(-13) \pm \sqrt{(-13)^2 - 4(1)(40)}}{2(1)} \\
 &= \frac{13 \pm \sqrt{169 - 160}}{2} \\
 &= \frac{13 \pm \sqrt{9}}{2} \\
 &= \frac{13 \pm 3}{2} \\
 &= 5 \text{ or } 8.
 \end{aligned}$$

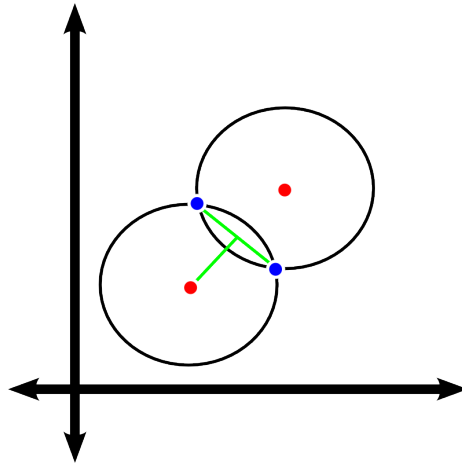
Using $x_0 = 5$ and our relation between x_0 and y_0 , we get:

$$\begin{aligned}
 y_0 &= x_0 - 4 \\
 y_0 &= 5 - 4 \\
 y_0 &= 1
 \end{aligned}$$

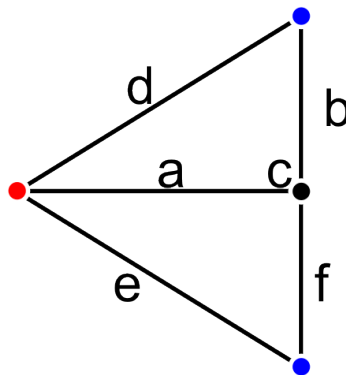
Now repeating for $x_0 = 8$ will yield $y_0 = 4$. Thus, the two circles of radius $\sqrt{5}$ and including the points $(7, 2)$ and $(6, 3)$ will have their centers located at the points $(5, 1)$ and $(8, 4)$ with the following equations:

$$\begin{aligned}
 (x - 5)^2 + (y - 1)^2 &= 5 \\
 (x - 8)^2 + (y - 4)^2 &= 5.
 \end{aligned}$$

Solution: Some people prefer to use a more geometric approach. To do that, we first highlight the steps we first have to draw some additional lines in the diagram:



The first line is a line joining the two points we know and hence is a *chord* of both circles. The second line is a line joining the midpoint of the chord and the center of one circle. Let's draw a quick sketch of the situation:



To find the location of the center of the circle, we need a few things:

1. To find the midpoint of the bisector between the two known points on the circle.
2. To show that the line between the center of a circle to the midpoint of the bisector forms a right angle (Angle c). This is a necessary step. There is no reason for two lines to meet at right angle just because it would be convenient for them to.
3. To find the distance between the midpoint and center (length a).
4. The equation of the line that length a is representing.
5. Combine all the information.

We will now tackle each step in order:

1. We know that the midpoint is at $(6.5, 2.5)$ due to the midpoint formula (See the textbook).
2. To show that c is a right angle, we make several observations about the two smaller triangles:

- Sides b and f are the same length (midpoint);
- Sides d and e are the same length (radius);
- Side a is common to both triangles.

Therefore, we can conclude that the two smaller triangles have the same side lengths and hence are congruent triangles. That means the two angles around the midpoint must be the same (ie 90° each). Thus we have two right angle triangles around the midpoint.

3. To find length a , we will apply the Pythagorean theorem with lengths b and d . But that means we need the length of b first. Using one of the given points and the midpoint, we have:

$$\begin{aligned} b &= \sqrt{(6.5 - 6)^2 + (2.5 - 3)^2} \\ &= \sqrt{(0.5)^2 + (-0.5)^2} \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

Now combining with the fact that d is the radius, we have

$$\begin{aligned} a^2 &= d^2 - b^2 \\ &= (\sqrt{5})^2 - \left(\frac{\sqrt{2}}{2}\right)^2 \\ a^2 &= \frac{9}{2} \\ a &= \frac{3\sqrt{2}}{2} \end{aligned}$$

That means the center of the circles are at a distance $\frac{3\sqrt{2}}{2}$ away from the midpoint.

4. We have established the line a represents is perpendicular to the bisector. So if we know the slope of the bisector, we can use the point slope formula to get the equation of the perpendicular line:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 3}{7 - 6} \\ &= -1 \end{aligned}$$

which means the perpendicular line has slope

$$m_{\perp} = 1$$

Finally, we can obtain an equation for the line perpendicular to the bisector.

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 2.5 &= 1(x - 6.5) \\ y &= x - 6.5 + 2.5 \\ y &= x - 4. \end{aligned}$$

5. Combining all this information, we have that the center of the circle is a point on the line $y = x - 4$ and is a distance $\frac{3\sqrt{2}}{2}$ away from the midpoint of the bisector. Applying the distance equation, we get that:

$$a = \sqrt{(x_c - 6.5)^2 + (y_c - 2.5)^2}$$

where a is the distance between the center of the circle at (x_c, y_c) and the midpoint of the bisector at $(6.5, 2.5)$

$$\left(\frac{3\sqrt{2}}{2}\right)^2 = (x_c - 6.5)^2 + (y_c - 2.5)^2$$

we know that the center is on the line $y = x - 4$, so that gives us a relation between y_c and x_c .

$$\begin{aligned}\frac{9}{2} &= (x_c - 6.5)^2 + ((x_c - 4) - 2.5)^2 \\ \frac{9}{2} &= (x_c - 6.5)^2 + (x_c - 6.5)^2 \\ \frac{9}{2} &= 2(x_c - 6.5)^2 \\ \frac{9}{4} &= (x_c - 6.5)^2 \\ \pm\frac{3}{2} &= x_c - 6.5\end{aligned}$$

which means

$$x_c = 5, 8$$

We make use of the fact that $y_c = x_c - 4$ to get that the centers are at $(5, 1)$ and $(8, 4)$. This means the equation of the circles are:

$$\begin{aligned}(x - 5)^2 + (y - 1)^2 &= 5 \\ (x - 8)^2 + (y - 4)^2 &= 5.\end{aligned}$$

(Which agrees with the solutions obtained above).

2. A function f has a *fixed point* at x_0 if $f(x_0) = x_0$. For example, the function $f(x) = \sqrt{x}$ has a fixed point at 1.
- (a) Give an example of a function with two fixed points.

Solution: There are infinitely many functions with two fixed points. For example the function $f(x) = x$ contains a fixed point for every possible x . So one way to think about this problem is to find a function that intersects with $f(x) = x$ at least twice (or exactly twice). If we think about this a little bit more, we realise that to get two fixed point, we really need a function that starts above the $f(x) = x$ line, goes below it, then above it again. Or starts below it, goes above it, then back below. Since the

function $f(x) = x$ has slope one, we really need the following two conditions:

- The functions goes down then up, or up then down (relative to $f(x) = x$).
- on the up bits, the slope has to be greater than one, otherwise it won't "catch up" to $f(x) = x$.

Two examples of this might be:

$$g(x) = x^2$$

$$g(x) = \sqrt{x}$$

and also all vertical dilations of the above functions will also have two fixed points. But we can get a little fancy and look at the following:

$$g(x) = 2|x - 1|$$

$$g(x) = -2|x + 1|.$$

If one wants to get really fancy, we can also start looking at piecewise functions such as:

$$g(x) = \begin{cases} 0 & x < 1 \\ 1 & x \geq 1 \end{cases} \qquad g(x) = \begin{cases} 1 - x & x < 1 \\ 5x - 5 & x \geq 1 \end{cases}$$

but we don't need it to be defined on all \mathbb{R} either, so something like:

$$g(x) = \begin{cases} -5 & x < 0 \\ 20 & x > 10 \end{cases} \qquad g(x) = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \end{cases}$$

will all work. There are literally infinite solutions.

- (b) Come up with a function that has exactly two fixed points at 1 and 3.

Solution: We can start with any of the functions from the previous part and then transform them so that the two fixed points are at $x = 1$ and $x = 3$. For example, starting with $f(x) = x^2$, we note that the fixed points are at $x = 0, 1$. We want to move the fixed point at 0 to 1 and the fixed point at 1 to 3. We can start with shifting the graph right by 1 and then up by one. This will give me:

$$f(x) = x^2$$

$$f(x - 1) = (x - 1)^2 \qquad \text{Right shift by 1}$$

$$f(x - 1) + 1 = (x - 1)^2 + 1 \qquad \text{Up shift by 1}$$

This will give me a fixed point at 1. To get the second one, we introduce a variable k to control the steepness of the curve.

$$k \cdot f(x - 1) + 1 = k(x - 1)^2 + 1 \qquad \text{Vertical dilation by } k$$

We can then adjust k such that the second fixed point is at $x = 3$. That is:

$$\begin{aligned}k(3 - 1)^2 + 1 &= 3 \\k(2)^2 &= 3 - 1 \\4k &= 2 \\k &= \frac{1}{2}\end{aligned}$$

Thus, the equation

$$g(x) = \frac{1}{2}(x - 1)^2 + 1$$

will have two fixed points at $x = 1, 3$ respectively.

Using a similar technique, we can get that

$$g(x) = \sqrt{2} \cdot \sqrt{x - 1} + 1$$

will also have fixed points at $x = 1, 3$. There are a infinite number of solutions as per the previous question. As a non-standard solution, we can even have:

$$g(x) = \begin{cases} 1 & x = 1 \\ 3 & x = 3 \end{cases}$$

We then have to show that these are the only fixed point. The simplest way of justifying this is to look at $f(x) - x$ and reason why that function only intersects the x -axis twice. For example,

$$g(x) - x = \frac{1}{2}(x - 1)^2 + 1 - x$$

is a quadratic and so has at most two x -intercepts.

Solution: An alternate Solution: We can hunt for a quadratic in a more algebraic manner. Let's define the quadratic we want to be $g(x) = ax^2 + bx + c$ for some coefficients a, b and c . At the points $x_0 = 1, 3$,

$$\begin{aligned}g(x_0) &= x_0 \\g(x_0) - x_0 &= 0.\end{aligned}$$

Further, we want the above equation to be true only at $x_0 = 1, 3$ and at other values for x , we have $g(x) - x \neq 0$. One function that will accomplish this is:

$$g(x) - x = (x - 1)(x - 3).$$

The right hand side will be yield 0 only when $x = 1, 3$, which is exactly what we want. Now we expand both sides to get:

$$\begin{aligned}(ax^2 + bx + c) - x &= x^2 - 4x + 3 \\ax^2 + (b - 1)x + c &= x^2 - 4x + 3\end{aligned}$$

By matching coefficients, we get

$$a = 1$$

$$b = -3$$

$$c = 3$$

After substitution, we get that the quadratic $g(x) = x^2 - 3x + 3$ will have two fixed points at $x = 1, 3$. We should note that while $(x - 1)(x - 3)$ will work, any multiple of that will also work. For example, solving for

$$g(x) - x = -4(x - 1)(x - 3)$$

will yield a different (but also correct) quadratic with fixed points at $x = 1, 3$. A similar justification is required to show that these are the only fixed points. Again, we can look at $g(x) - x$ and see that it is a quadratic with x -intercepts at $x = 1, 3$ only.

3. Consider two non-vertical perpendicular lines passing through the origin. That is, the lines meet at a right angle. Prove that the slopes of the two lines are negative reciprocals of each other.

Solution: There is a proof of this result in the text book in Section 0.2 and the relevant figure is Fig. 15. Consider two non-vertical lines passing through the origin. Denote the slope of the first by m_1 and the slope of the second by m_2 . Let (x_1, y_1) be a point on the first line (other than the origin) and (x_2, y_2) a point on the second line (other than the origin). Using the formula for slope we obtain,

$$m_1 = \frac{y_1}{x_1}, \quad m_2 = \frac{y_2}{x_2}. \quad (1)$$

We seek a relationship between m_1 and m_2 via x_1, x_2, y_1 and y_2 . Suppose now that these lines are perpendicular. In this way the triangle with vertices $(0, 0)$, (x_1, y_1) and (x_2, y_2) is a right triangle. Let d_1 be the distance between $(0, 0)$ and (x_1, y_1) . Let d_2 be the distance between $(0, 0)$ and (x_2, y_2) . Let d_3 be the distance between (x_1, y_1) and (x_2, y_2) . Lengths d_1, d_2 and d_3 are the sides of a right angle triangle. An application of the Pythagorean Theorem yields,

$$d_1^2 + d_2^2 = d_3^2. \quad (2)$$

Moreover, we can find each of these lengths by means of the distance formula,

$$d_1^2 = x_1^2 + y_1^2,$$

$$d_2^2 = x_2^2 + y_2^2,$$

$$d_3^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Substituting to (2) gives,

$$x_1^2 + y_1^2 + x_2^2 + y_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

or,

$$\begin{aligned}x_1^2 + y_1^2 + x_2^2 + y_2^2 &= x_2^2 - 2x_1x_2 + x_1^2 + y_2^2 - 2y_1y_2 + y_1^2 \\0 &= -2x_1x_2 - 2y_1y_2 \\y_1y_2 &= -x_1x_2 \\ \frac{y_2}{x_1} &= -\frac{x_2}{y_1}\end{aligned}$$

after some expansion, simplification and rearrangement. Observe, that (1) gives,

$$m_2 = \frac{y_2}{x_1} = -\frac{x_2}{y_1} = -\frac{1}{m_1}.$$

Hence, the slopes of the two lines are negative reciprocals, the desired result.

Alternatively, let θ_1 be the angle line one makes with the x axis and let θ_2 be the angle line two makes with the x axis. Using some trigonometry we can show that $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$ are the slopes of the two lines. We have assumed that these lines are perpendicular so $\theta_2 = \theta_1 + \pi$, (in radians). Note the following trigonometric identity,

$$\tan(\theta + \pi/2) = -\frac{\cos \theta}{\sin \theta}.$$

We apply the above to θ_2 to see,

$$\begin{aligned}m_2 &= \tan \theta_2 \\ &= \tan(\theta_1 + \pi/2) \\ &= -\frac{\cos \theta_1}{\sin \theta_1} \\ &= -\left(\frac{\cos \theta}{\sin \theta}\right)^{-1} \\ &= -(\tan \theta_1)^{-1} \\ &= -\frac{1}{m_1}.\end{aligned}$$

This is the required result and so we are done.