

**Final Examination — December 5<sup>th</sup> 2014****Duration: 2.5 hours***This test has 11 questions on 20 pages, for a total of 100 points.*

- Read all the questions carefully before starting to work.
- Q1 is a multiple choice question; circle the correct answer.
- Q2-Q5 are short-answer questions; put your answer in the boxes provided.
- All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: \_\_\_\_\_ Last Name: \_\_\_\_\_

Student-No: \_\_\_\_\_ Section: \_\_\_\_\_

Signature: \_\_\_\_\_

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	10	10	12	9	9	7	7	14	7	6	9	100
Score:												

**Student Conduct during Examinations**

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
  - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
  - (iii) purposely viewing the written papers of other examination candidates;
  - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

**Multiple choice questions:****10 marks**1. Each part is worth 1 mark. **Circle the correct answer.**

Note — page 4 is blank for your rough work for Q1.

(a) The limit  $\lim_{x \rightarrow 1} \log(x)$ 

- A. does not exist
- B. = 0
- C. = 1
- D. =  $e$

(b) The limit  $\lim_{x \rightarrow 0^-} \frac{1}{x}$ 

- A. does not exist
- B. does not exist but “=”  $+\infty$
- C. does not exist but “=”  $-\infty$
- D. = 0

(c) The slope of the line tangent to the graph  $f(x) = x^2 + 4\sqrt{x}$  at  $x = 4$  is

- A. Undefined
- B. 6
- C. 7
- D. 9

(d) If  $f(x) = \cos x$  then  $f'(x)$  is

- A.  $\sin x$
- B.  $-\sin x$
- C.  $-\cos x$
- D.  $\sec x$

(e) If  $g(x) = \arcsin x$  then  $\frac{dg}{dx}$  is

- A.  $\cos x$
- B.  $\frac{1}{\cos x}$
- C.  $\frac{1}{\sqrt{1-x^2}}$
- D.  $\frac{1}{1+x^2}$

(f)  $\arcsin\left(-\frac{1}{2}\right)$  is

- A.  $-\frac{\pi}{3}$
- B.  $-\frac{\pi}{6}$
- C.  $\frac{\pi}{6}$
- D.  $\frac{\pi}{3}$

(g) Let  $g(x) = \log(x)$ . The line tangent to  $g(x)$  at the point  $(a, \log(a)) \dots$

- A. lies above the function for all values of  $a > 0$ .
- B. lies below the function for all values of  $a > 0$ .
- C. lies above the function for  $a > 1$  and below the function for  $0 < a < 1$ .
- D. lies above the function for  $a < 0$  and below the function for  $a > 0$ .

(h) The function  $f(x) = \sin x$  has

- A. a discontinuity at  $x = 0$
- B. a local maximum at  $x = 0$
- C. a local minimum at  $x = 0$
- D. a point of inflection at  $x = 0$

(i) The function  $g(x) = \frac{1}{\tan x}$  has

- A. a discontinuity at  $x = 0$
- B. a local maximum at  $x = 0$
- C. a local minimum at  $x = 0$
- D. a point of inflection at  $x = 0$

(j) The function  $h(x) = \frac{1}{\cos x}$  has

- A. a discontinuity at  $x = 0$
- B. a local maximum at  $x = 0$
- C. a local minimum at  $x = 0$
- D. a point of inflection at  $x = 0$

*This page has been left blank for your workings and solutions.*

**Short-answer questions:** Put your answer in the box provided. Full marks will be given for a correct answer placed in the box, while part marks may be given if workings are shown. Please simplify your answers.

2. Each part of this question is worth 2 marks.

2 marks

(a) Compute  $\lim_{t \rightarrow 0} \arcsin(\cos(t))$ . **Simplify your answer.**

Answer:

2 marks

(b) Compute  $\lim_{x \rightarrow -3^+} \frac{x+2}{x+3}$ .

Answer:

2 marks

(c) Find the domain of  $f(x) = \log(2 - \log(x))$ .

**Note:**  $\log(x)$  is the natural logarithmic function, also denoted  $\ln(x)$ .

Answer:

2 marks

(d) Compute  $\lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 + 1} - x}$ .

Answer:

2 marks

(e) Compute  $\lim_{x \rightarrow +\infty} e^{\left(\frac{\cos(x^2)}{x}\right)}$ .

Answer:

3. Each part of this question is worth 2 marks.

2 marks

(a) Let  $f(x) = e^x \cdot \cos(\pi x)$ . Find  $f'(x)$ .

Answer:

2 marks

(b) Let  $y = \log(\sin(\log x))$ . Find  $\frac{dy}{dx}$ .

**Note:**  $\log(x)$  is the natural logarithmic function, also denoted  $\ln(x)$ .

Answer:

2 marks

(c) Let  $y = x^{\log x}$ . Find  $\frac{dy}{dx}$ .

**Note:**  $\log(x)$  is the natural logarithmic function, also denoted  $\ln(x)$ .

Answer:

2 marks

(d) Find  $\lim_{t \rightarrow 0} \frac{\sqrt{t+1} - e^t}{t}$ .

Answer:

2 marks

- (e) Air is being pumped into a spherical balloon at a rate of  $100 \text{ cm}^3/\text{s}$ . How fast is the radius of the balloon increasing when the diameter is 40 cm?

Answer:

2 marks

(f) Compute  $\theta = \arcsin \left( \sin \left( \frac{31\pi}{11} \right) \right)$ .

Answer:



4. Each part of this question is worth 3 marks.

3 marks

(a) Compute

$$\lim_{x \rightarrow 0} \frac{\cos x - e^{x^2}}{x^2}$$

Answer:

3 marks

(b) Let  $f(x)$  be a function differentiable at  $x = 1$  and let  $g(x) = \frac{f(x)}{x^2}$ . The line tangent to the curve  $y = f(x)$  at  $x = 1$  has slope 3 while the line tangent to the curve  $y = g(x)$  at  $x = 1$  has slope 4. What is  $f(1)$ ?

Answer:

**3 marks**

(c) Consider the function

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Use a linear approximation to estimate  $f\left(\frac{1}{10}\right)$ .

Answer:

5. Each part is worth 3 marks.

3 marks

(a) Find  $a$  such that

$$f(x) = \sin(ax) - x^2 + 2x + 3$$

has a critical point at  $x = 0$ .

Answer:

3 marks

(b) Find the **coordinates** of the global minimum of the function

$$f(x) = x^3 - 6x + 4$$

on the interval  $[0, 3]$ .

Answer:

3 marks
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(c) If  $f'(x) = 3\sqrt{x} - \frac{5}{x^2}$  and  $f(1) = -4$ , find  $f(x)$ .

Answer:
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**Full-solution problems:** Justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions**.

4 marks

6. (a) Let  $g(x) = \frac{x}{2+3x}$ . Compute  $g'(x)$  using the definition of the derivative.

**No marks will be given for use of derivative rules**, but you may use them to check your answer.

3 marks
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(b) **Without using L'Hôpital's rule** compute the limit

$$\lim_{x \rightarrow +\infty} \left( \sqrt{x^2 + 2x} - \sqrt{x^2 - 2x} \right)$$

**Solutions using L'Hôpital's rule will get zero marks.**

7. Particle 1 moves on the  $y$ -axis starting at the point  $(0, 6)$  and travels towards origin with constant speed of 2 units per second. Particle 2 moves on the  $x$ -axis starting at the origin and travels in the positive  $x$ -direction with constant speed of 1 unit per second.

3 marks

- (a) Write a formula for the distance between the particles as a function of time.

4 marks

- (b) At what moment in the first 3 seconds is the distance minimised?

8. Let  $f(x) = \frac{\log(x)}{\sqrt{x}}$ .

**Note**  $\log(x)$  is the natural logarithmic function, also denoted  $\ln(x)$ .

1 mark

(a) Find the domain of  $f$ .

Answer:

2 marks

(b) Find the  $x$ -intercepts and  $y$ -intercepts of the function.

Answer:

2 marks

(c) Find the horizontal asymptotes of the graph of  $f(x)$ , if they exist. Justify your answer.

Answer:

2 marks

(d) Find the vertical asymptotes of the graph of  $f(x)$ , if they exist. Justify your answer.

Answer:



4 marks
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- (e) Find the intervals where  $f(x)$  is increasing, and the intervals where  $f(x)$  is decreasing. Find the **coordinates** of local maximum and minimum if they exist.

3 marks
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- (f) The second derivative of  $f(x)$  is

$$f''(x) = \frac{3 \log(x) - 8}{4x^{5/2}}.$$

Determine the intervals on which  $f$  is concave up and the intervals on which  $f$  is concave down. Find the **coordinates** of any inflection points if they exist.

- (g) **There is no (g) — you do not need to sketch the graph.**

9. Consider the equation  $x^4 + 4x + q = 0$ , where  $q$  is some real number.

2 marks

(a) Find a value of  $q$  so that the equation has at least two solutions.

5 marks

(b) Now let  $q$  be any real number. Prove that the equation has at most two solutions.

**6 marks** 10. Let  $a$  be a real number, and let

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x^a \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

For what values  $a$  is the function  $f(x)$  continuous at  $x = 0$ ? **Justify your answer.**

Answer:

4 marks
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11. (a) Let  $f(x) = e^x - e^{-x}$ .Find the global maximum and minimum of  $f(x)$  on the interval  $[-1, 1]$ .

5 marks
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(b) Now let  $g(x) = e^x + e^{-x}$ .Your friend uses a degree-2 Taylor polynomial about  $a = 0$  to approximate  $g(1)$ .Show that the absolute error in their approximation is no greater than  $4/9$ .