

Midterm Test 1 Duration: 60 minutes*This test has 6 questions on 11 pages, for a total of 43 points.*

- Read all the questions carefully before starting to work.
- Q1 and Q2 are short-answer questions; put your answer in the boxes provided.
- All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: _____ Last Name: _____

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	5	6	Total
Points:	9	12	5	7	5	5	43
Score:							

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Short-Answer Questions. Questions 1 and 2 are short-answer questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box. Show your work also, for part marks. Each part is worth 3 marks, but not all parts are of equal difficulty. **Simplify your answers as much as possible in Questions 1 and 2.**

9 marks

1. Determine whether each of the following limits exists, and find the value if they do. If a limit below does not exist, determine whether it “equals” ∞ , $-\infty$, or neither.

(a)

$$\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x^2 - 9}$$

Answer: $\frac{5}{6}$ **Solution:**

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x^2 - 9} &= \lim_{x \rightarrow 3} \frac{(x-3)(x+2)}{(x-3)(x+3)} && \text{factor top/bottom} \\ &= \lim_{x \rightarrow 3} \frac{(x+2)}{(x+3)} && \text{simplify} \\ &= \frac{3+2}{3+3} = \frac{5}{6} && \text{substitute/simplify} \end{aligned}$$

(b)

$$\lim_{x \rightarrow -2} \frac{\sqrt{x^2 + 5} - 3}{2 + x}$$

Answer: $-\frac{2}{3}$

Solution: When $x \rightarrow -2$ both numerator and denominator go to zero — something cancels.

$$\begin{aligned} \frac{\sqrt{x^2 + 5} - 3}{2 + x} &= \frac{\sqrt{x^2 + 5} - 3}{2 + x} \cdot \frac{\sqrt{x^2 + 5} + 3}{\sqrt{x^2 + 5} + 3} && \text{multiply by conj} \\ &= \frac{x^2 + 5 - 9}{(x+2)(\sqrt{x^2 + 5} + 3)} = \frac{x^2 - 4}{(x+2)(\sqrt{x^2 + 5} + 3)} && \text{clean up} \\ &= \frac{(x+2)(x-2)}{(x+2)(\sqrt{x^2 + 5} + 3)} && \text{factor} \\ &= \frac{x-2}{\sqrt{x^2 + 5} + 3} && \text{cancel} \end{aligned}$$

Hence

$$\begin{aligned} \lim_{x \rightarrow -2} \frac{\sqrt{x^2 + 5} - 3}{2 + x} &= \lim_{x \rightarrow -2} \frac{x-2}{\sqrt{x^2 + 5} + 3} \\ &= \frac{-4}{\sqrt{4+5} + 3} = \frac{-4}{\sqrt{9} + 3} \\ &= -\frac{4}{6} = -\frac{2}{3} \end{aligned}$$

(c)

$$\lim_{h \rightarrow 0} \left(\frac{1}{h} - \frac{1}{|h|} \right)$$

Answer: Does not exist

Solution: The left and right limits are different:

$$\lim_{h \rightarrow 0^+} \frac{1}{h} - \frac{1}{|h|} = \lim_{h \rightarrow 0^+} \frac{1}{h} - \frac{1}{h} = 0 \quad \text{from right}$$

$$\lim_{h \rightarrow 0^-} \frac{1}{h} - \frac{1}{|h|} = \lim_{h \rightarrow 0^-} \frac{1}{h} + \frac{1}{h} = -\infty \quad \text{from left}$$

Thus the limit does not exist.

12 marks

2. (a) If
- $f(x) = e^x + x^3 + e^\pi$
- , find
- $f'(x)$
- .

$$\text{Answer: } f' = e^x + 3x^2$$

Solution: Since $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}x^n = nx^{n-1}$ and $\frac{d}{dx}c = 0$ (where c is a constant):

$$f'(x) = e^x + 3x^2$$

- (b) Find the equation of the line tangent to
- $f(x) = e^{\cos x}$
- at the point
- $x = \frac{\pi}{2}$
- .

$$\text{Answer: } y - 1 = -(x - \pi/2)$$

Solution: We use chain rule to find the slope

$$\begin{aligned} f'(x) &= e^{\cos x}(-\sin x) \\ f'(\pi/2) &= -e^0 = -1 \end{aligned}$$

As well the point $(\pi/2, f(\pi/2)) = (\pi/2, 1)$ must lie on the line. Using the equation of the line in point slope form we get

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 1 &= -(x - \pi/2) \end{aligned}$$

(c) Let $h(x) = \sqrt{x+1} + \tan x$. Determine where $h(x)$ is differentiable.

Answer: all x real with $x > -1$ and $x \neq \pi/2 + n\pi$ where n is an integer

Solution:

- $\tan x$ is differentiable on its domain: $x \neq \pi/2 + n\pi$ where n is an integer
- $\sqrt{x+1}$ is differentiable on its domain ($x \geq -1$) except at $x = -1$ where it has a vertical tangent line.

(d) Let

$$f(x) = \frac{xg(x)}{\sin x}$$

where $g(\pi/2) = \pi$ and $g'(\pi/2) = 2$. Compute $f'(\pi/2)$.

Answer: $f'(\pi/2) = 2\pi$

Solution: We use the product and quotient rules

$$\begin{aligned} f'(x) &= \frac{(xg'(x) + g(x)) \sin x - xg(x) \cos x}{\sin^2 x} \\ f'(\pi/2) &= \frac{(\frac{\pi}{2}g'(\pi/2) + g(\pi/2)) \sin(\pi/2) - \frac{\pi}{2}g(\pi/2) \cos(\pi/2)}{\sin^2(\pi/2)} \\ f'(\pi/2) &= \frac{(\pi + \pi)(1) - 0}{1} = 2\pi \end{aligned}$$

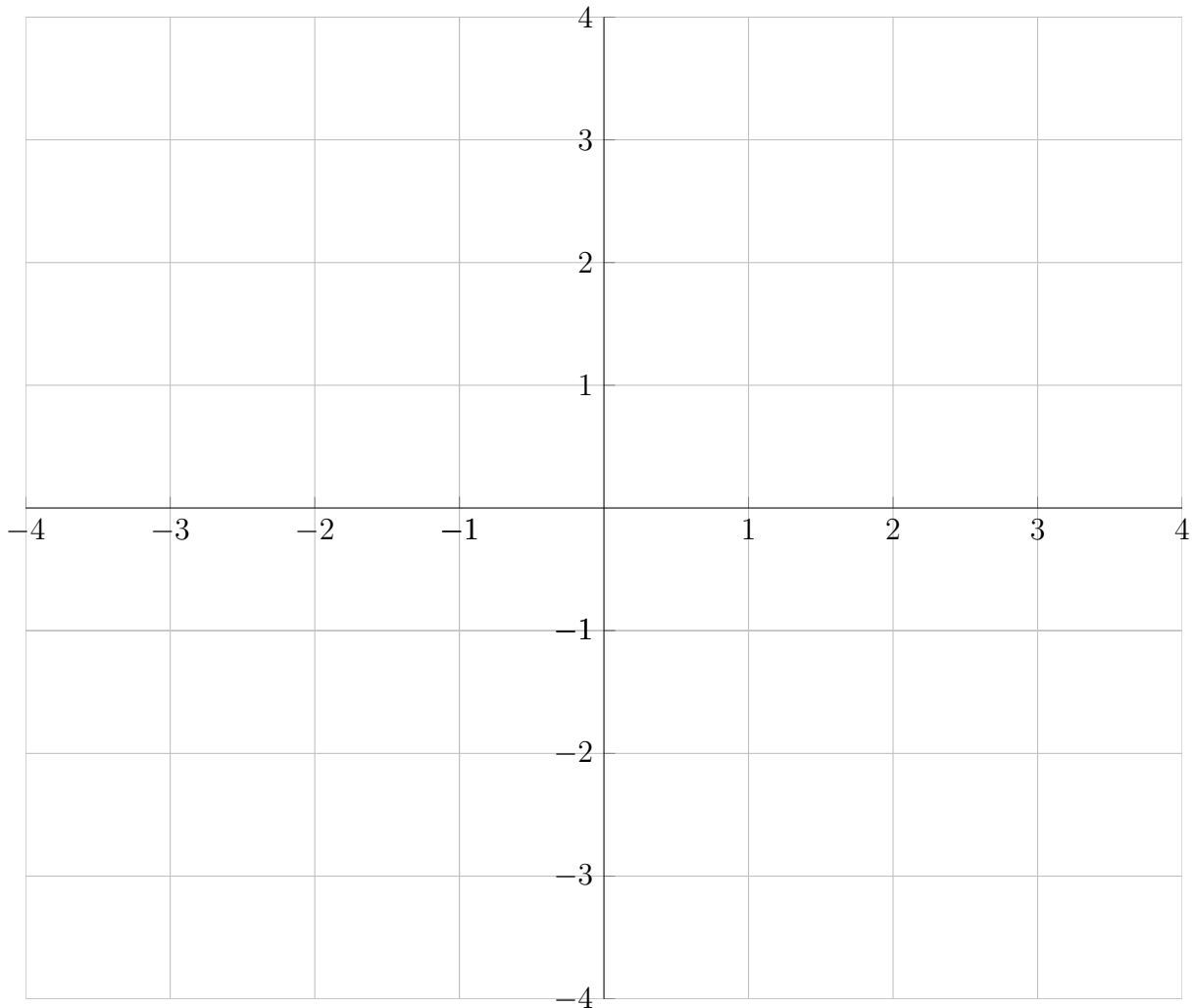
Full-Solution Problems. In questions 3–6, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions**.

5 marks

3. Sketch the graph of a function satisfying the following properties:

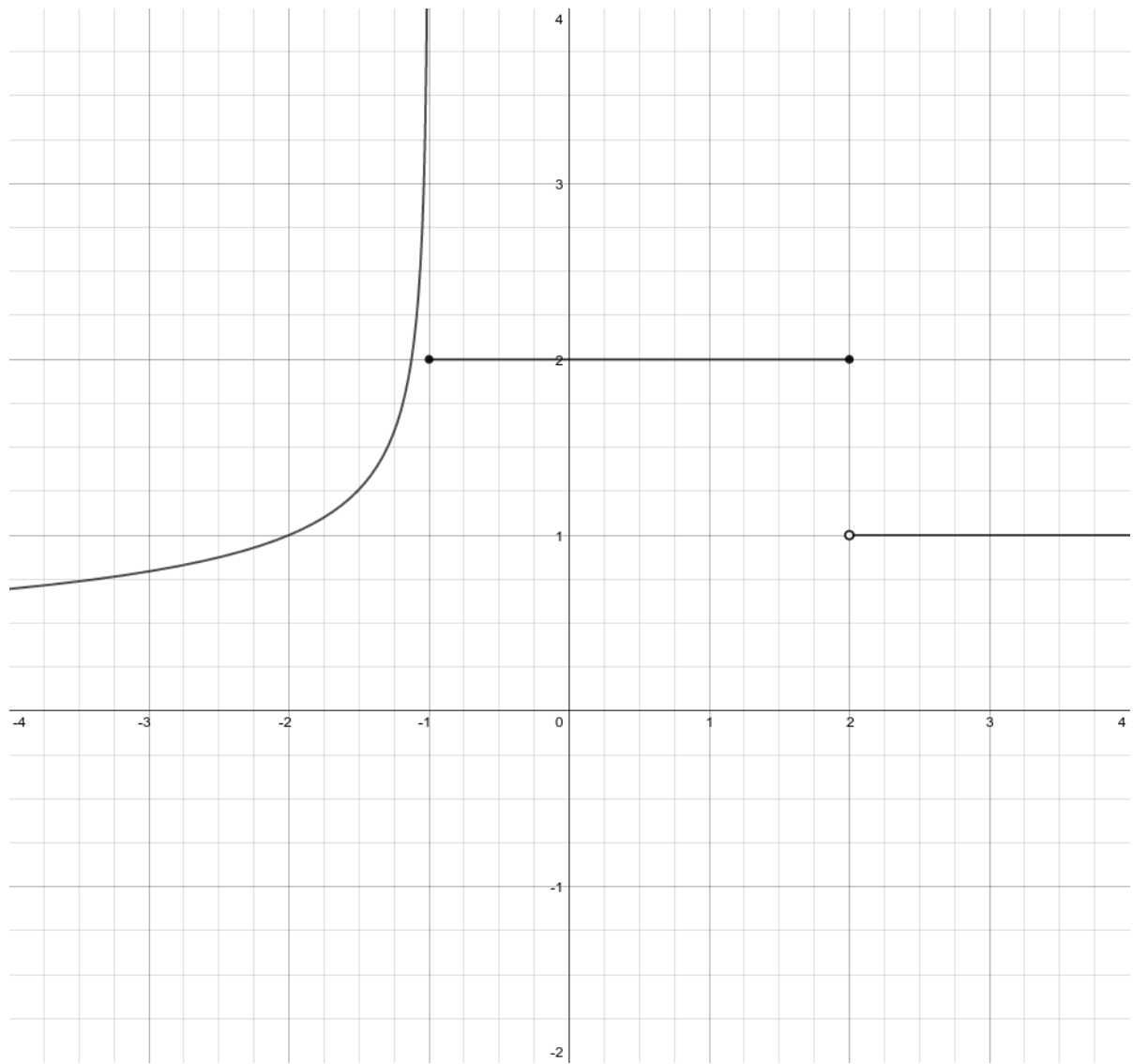
- The domain of $f(x)$ is $[-4, 4]$
- $f(x)$ is left continuous at $x = 2$
- $f(x)$ is not right continuous at $x = 2$
- $f(x)$ has a vertical asymptote at $x = -1$
- $\lim_{x \rightarrow -1^+} f(x) = 2$

You do not need to find an equation for your function. Use the axes below.



There is another set of axes on the following page (in case you ruin the first one).

Solution: There are many functions that satisfy the conditions. Here is an example of one.



Marking scheme:

- 1 mark for each condition your graph meets.

4. Let $f(x) = \frac{\sqrt{2x^2 + 3}}{x + 1}$

4 marks

(a) Determine the horizontal asymptotes of the graph $y = f(x)$

Answer: As $x \rightarrow +\infty$, $f \rightarrow \sqrt{2}$
and as $x \rightarrow -\infty$, $f \rightarrow -\sqrt{2}$.

Solution: Assume $x > 0$ then

$$\begin{aligned} \frac{\sqrt{2x^2 + 3}}{x + 1} &= \frac{\sqrt{x^2} \sqrt{2 + 3/x^2}}{x + 1} \\ &= \frac{x \sqrt{2 + 3/x^2}}{x(1 + 1/x)} \\ &= \frac{\sqrt{2 + 3/x^2}}{(1 + 1/x)} \end{aligned}$$

so as $x \rightarrow +\infty$

$$\lim_{x \rightarrow +\infty} f(x) = \frac{\sqrt{2}}{1} = \sqrt{2}$$

Now assume $x < 0$ then

$$\begin{aligned} \frac{\sqrt{2x^2 + 3}}{x + 1} &= \frac{\sqrt{x^2} \sqrt{2 + 3/x^2}}{x + 1} \\ &= \frac{-x \sqrt{2 + 3/x^2}}{x(1 + 1/x)} \\ &= -\frac{\sqrt{2 + 3/x^2}}{(1 + 1/x)} \end{aligned}$$

so as $x \rightarrow -\infty$

$$\lim_{x \rightarrow -\infty} f(x) = -\frac{\sqrt{2}}{1} = -\sqrt{2}$$

Marking scheme:

- Correct simplification when $x > 0$ — 1 mark
- Correct positive limit — 1 mark
- Correct simplification when $x < 0$ — 1 mark
- Correct negative limit — 1 mark

3 marks

(b) Determine the vertical asymptote(s) of the graph $y = f(x)$. For each vertical asymptote $x = a$, determine whether each of the one-sided limits “equals” $+\infty$ or $-\infty$ as x approaches a .

Solution:

- Since the numerator and denominator are continuous functions for all x (in their domains), the only possible asymptote occurs when the denominator is zero. Namely where $x = -1$.
- As $x \rightarrow -1$ the numerator goes to $\sqrt{2+3} = \sqrt{5}$.
- As $x \rightarrow -1^+$ the denominator goes to zero but is positive. Hence

$$\lim_{x \rightarrow -1^+} f(x) = +\infty$$

- As $x \rightarrow -1^-$ the denominator goes to zero but is negative. Hence

$$\lim_{x \rightarrow -1^-} f(x) = -\infty$$

Marking scheme:

- Deducing asymptote only when denominator is zero — 1 mark
- Correct left-limit — 1 mark
- Correct right-limit — 1 mark

5 marks

5. Show that the following equation has at least one solution:

$$\cos x - \tan x = 0$$

Solution: We use the intermediate value theorem.

- Let $f(x) = \cos x - \tan x$. The function $\cos x$ is continuous everywhere and $\tan x$ is continuous on its domain so in particular it is continuous on $(-\pi/2, \pi/2)$.
- Compute $f(0) = \cos 0 - \tan 0 = 1 > 0$
- Compute $f(\pi/4) = \cos(\pi/4) - \tan(\pi/4) = 1/\sqrt{2} - 1 < 0$ since $\sqrt{2} > 1$
- By the IVT since $f(x)$ is continuous on $[0, \pi/4]$ and is positive at $x = 0$ and negative at $x = \pi/4$, there exists some $0 < c < \pi/4$ so that $f(c) = 0$.

Marking scheme:

- Explaining function continuous — 2 marks
- Finding negative value — 1 mark
- Finding two positive values — 1 mark
- Invoking IVT — 1 marks

5 marks

6. Let $g(x)$ be a function satisfying

$$3x \leq g(x) \leq x^3 - 3x + 4$$

for all x . Now let

$$f(x) = \begin{cases} \sqrt{3-x} & x \leq 2 \\ g(x) - c & x > 2 \end{cases}$$

where c is a constant. Find the value of c that makes $f(x)$ continuous at $x = 2$. Ensure that your answer is fully justified; unjustified answers will not receive credit.

Solution: In order for f to be continuous at $x = 2$, the left and right limits must agree and be equal to the value of the function.

- At $x = 2$, $f(x) = f(2) = \sqrt{3-2} = \sqrt{1} = 1$.
- The left-limit is

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \sqrt{3-x} = 1$$

which is equal to $f(2)$.

- For the right limit

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} g(x) - c$$

To find this limit we will use the Squeeze Theorem on $g(x)$. We have that

$$3x \leq g(x) \leq x^3 - 3x + 4$$

As $x \rightarrow 2^+$ we have

$$\lim_{x \rightarrow 2^+} 3x = \lim_{x \rightarrow 2^+} x^3 - 3x + 4 = 6$$

Hence, by the squeeze theorem

$$\lim_{x \rightarrow 2^+} g(x) = 6$$

- In order to make f continuous we therefore need $1 = 6 - c$ so $c = 5$.

Marking scheme:

- Realising need left-right limits to agree — 1 marks.
- Easy limit — 1 mark
- Harder limit — 2 marks
- Correct value — 1 mark.