Midterm Test 1 Duration: 60 minutes

*This test has **6 questions** on **11 pages**, for a total of 43 points.*

- Read all the questions carefully before starting to work.
- Q1 and Q2 are short-answer questions; put your answer in the boxes provided.
- All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ____________ Last Name: ____________

Student-No: ____________ Section: ____________

Signature: ____________

Question:	1	2	3	4	5	6	Total
Points:	9	12	5	7	5	5	43
Score:							

Short-Answer Questions. Questions 1 and 2 are short-answer questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box. Show your work also, for part marks. Each part is worth 3 marks, but not all parts are of equal difficulty. **Simplify your answers as much as possible in Questions 1 and 2.**

9 marks 1. Determine whether each of the following limits exists, and find the value if they do. If a limit below does not exist, determine whether it "equals" ∞, $-\infty$, or neither.

(a)

$$\lim_{x\to 3} \frac{x^2 - x - 6}{x^2 - 9}$$

Answer: $\frac{5}{6}$

Solution:

$$\begin{aligned}\lim_{x\to 3} \frac{x^2 - x - 6}{x^2 - 9} &= \lim_{x\to 3} \frac{(x-3)(x+2)}{(x-3)(x+3)} && \text{factor top/bottom}\\ &= \lim_{x\to 3} \frac{(x+2)}{(x+3)} && \text{simplify}\\ &= \frac{3+2}{3+3} = \frac{5}{6} && \text{substitute/simplify}\end{aligned}$$

(b)

$$\lim_{x\to -2} \frac{\sqrt{x^2+5} - 3}{2+x}$$

Answer: $-\frac{2}{3}$

Solution: When $x \to -2$ both numerator and denominator go to zero — something cancels.

$$\begin{aligned}\frac{\sqrt{x^2+5}-3}{2+x} &= \frac{\sqrt{x^2+5}-3}{2+x} \cdot \frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3} && \text{multiply by conj}\\ &= \frac{x^2+5-9}{(x+2)(\sqrt{x^2+5}+3)} = \frac{x^2-4}{(x+2)(\sqrt{x^2+5}+3)} && \text{clean up}\\ &= \frac{(x+2)(x-2)}{(x+2)(\sqrt{x^2+5}+3)} && \text{factor}\\ &= \frac{x-2}{\sqrt{x^2+5}+3} && \text{cancel}\end{aligned}$$

Hence

$$\begin{aligned}\lim_{x\to -2} \frac{\sqrt{x^2+5}-3}{2+x} &= \lim_{x\to -2} \frac{x-2}{\sqrt{x^2+5}+3}\\ &= \frac{-4}{\sqrt{4+5}+3} = \frac{-4}{\sqrt{9}+3}\\ &= -\frac{4}{6} = -\frac{2}{3}\end{aligned}$$

(c)

$$\lim_{h\to 0}\left(\frac{1}{h}-\frac{1}{|h|}\right)$$

Answer: Does not exist

Solution: The left and right limits are different:

$$\lim_{h\to 0^+}\frac{1}{h}-\frac{1}{|h|}=\lim_{h\to 0^+}\frac{1}{h}-\frac{1}{h}=0 \qquad \text{from right}$$

$$\lim_{h\to 0^-}\frac{1}{h}-\frac{1}{|h|}=\lim_{h\to 0^-}\frac{1}{h}+\frac{1}{h}=-\infty \qquad \text{from left}$$

Thus the limit does not exist.

12 marks 2. (a) If $f(x) = e^x + x^3 + e^\pi$, find $f'(x)$.

Answer: $f' = e^x + 3x^2$

Solution: Since $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$ and $\frac{\mathrm{d}}{\mathrm{d}x} x^n = nx^{n-1}$ and $\frac{\mathrm{d}}{\mathrm{d}x} c = 0$ (where c is a constant):

$$f'(x) = e^x + 3x^2$$

(b) Find the equation of the line tangent to $f(x) = e^{\cos x}$ at the point $x = \frac{\pi}{2}$.

Answer: $y - 1 = -(x - \pi/2)$

Solution: We use chain rule to find the slope

$$f'(x) = e^{\cos x}(-\sin x)$$
$$f'(\pi/2) = -e^0 = -1$$

As well the point $(\pi/2, f(\pi/2)) = (\pi/2, 1)$ must lie on the line. Using the equation of the line in point slope form we get

$$y - y_0 = m(x - x_0)$$
$$y - 1 = -(x - \pi/2)$$

(c) Let $h(x) = \sqrt{x+1} + \tan x$. Determine where $h(x)$ is differentiable.

Answer: all x real with $x > -1$ and $x \neq \pi/2 + n\pi$ where n is an integer

Solution:

- $\tan x$ is differentiable on its domain: $x \neq \pi/2 + n\pi$ where n is an integer
- $\sqrt{x+1}$ is differentiable on its domain $(x \geq -1)$ except at $x = -1$ where it has a vertical tangent line.

(d) Let

$$f(x) = \frac{xg(x)}{\sin x}$$

where $g(\pi/2) = \pi$ and $g'(\pi/2) = 2$. Compute $f'(\pi/2)$.

Answer: $f'(\pi/2) = 2\pi$

Solution: We use the product and quotient rules

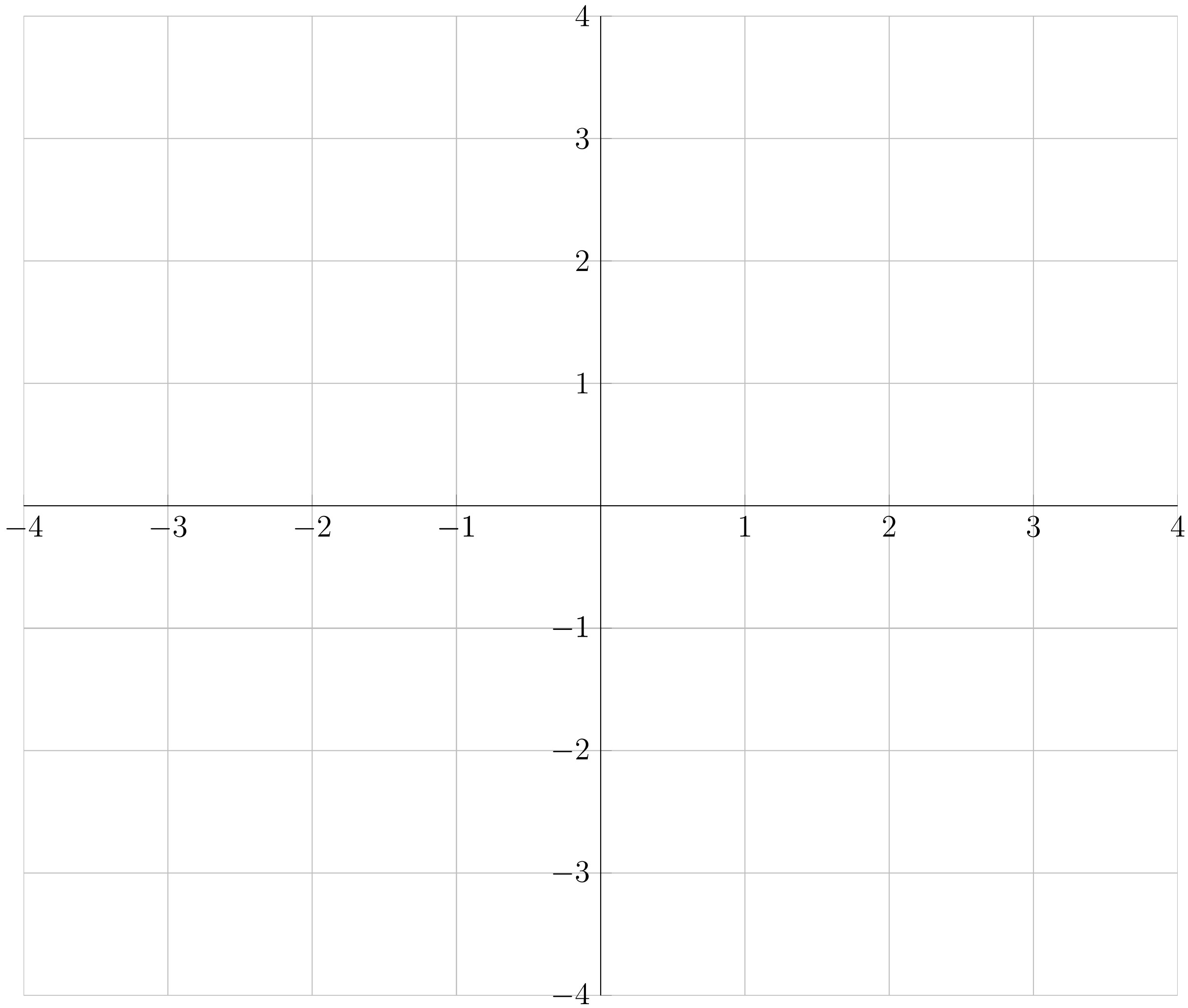
$$\begin{aligned} f'(x) &= \frac{(xg'(x) + g(x))\sin x - xg(x)\cos x}{\sin^2 x} \\ f'(\pi/2) &= \frac{(\frac{\pi}{2}g'(\pi/2) + g(\pi/2))\sin(\pi/2) - \frac{\pi}{2}g(\pi/2)\cos(\pi/2)}{\sin^2(\pi/2)} \\ f'(\pi/2) &= \frac{(\pi+\pi)(1) - 0}{1} = 2\pi \end{aligned}$$

Full-Solution Problems. In questions 3–6, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions**.

5 marks 3. Sketch the graph of a function satisfying the following properties:

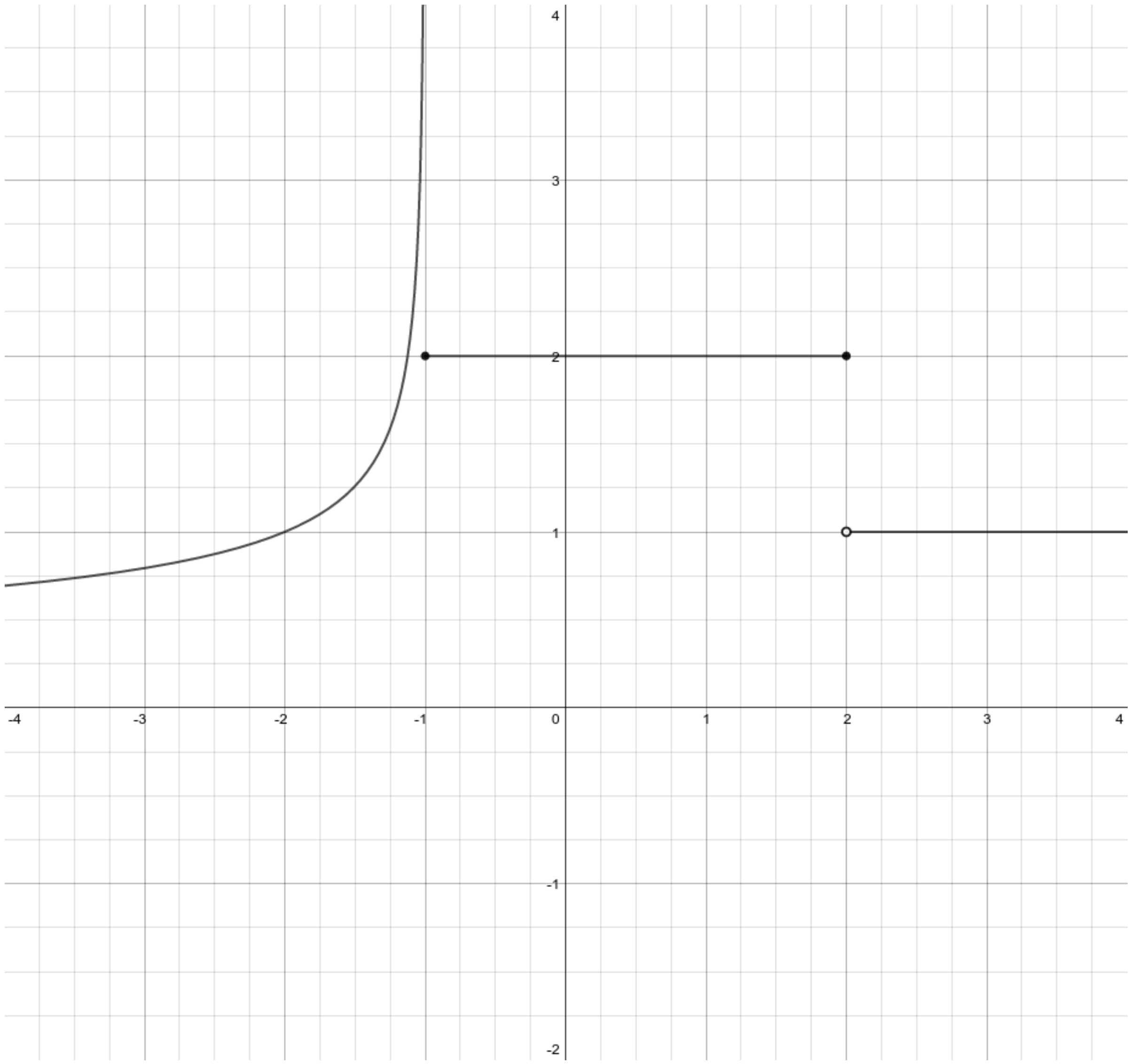
- The domain of $f(x)$ is $[-4, 4]$
- $f(x)$ is left continuous at $x = 2$
- $f(x)$ is not right continuous at $x = 2$
- $f(x)$ has a vertical asymptote at $x = -1$
- $\lim_{x \to -1^+} f(x) = 2$

You do not need to find an equation for your function. Use the axes below.



There is another set of a axes on the following page (in case you ruin the first one).

Solution: There are many functions that satisfy the conditions. Here is an example of one.



Marking scheme:

- 1 mark for each condition your graph meets.

4. Let $f(x) = \dfrac{\sqrt{2x^2+3}}{x+1}$

4 marks (a) Determine the horizontal asymptotes of the graph $y = f(x)$

Answer: As $x \to +\infty, f \to \sqrt{2}$ and as $x \to -\infty, f \to -\sqrt{2}$.

Solution: Assume $x > 0$ then

$$\begin{aligned}\frac{\sqrt{2x^2+3}}{x+1} &= \frac{\sqrt{x^2}\sqrt{2+3/x^2}}{x+1} \\ &= \frac{x\sqrt{2+3/x^2}}{x(1+1/x)} \\ &= \frac{\sqrt{2+3/x^2}}{(1+1/x)}\end{aligned}$$

so as $x \to +\infty$

$$\lim_{x\to+\infty} f(x) = \frac{\sqrt{2}}{1} = \sqrt{2}$$

Now assume $x < 0$ then

$$\begin{aligned}\frac{\sqrt{2x^2+3}}{x+1} &= \frac{\sqrt{x^2}\sqrt{2+3/x^2}}{x+1} \\ &= \frac{-x\sqrt{2+3/x^2}}{x(1+1/x)} \\ &= -\frac{\sqrt{2+3/x^2}}{(1+1/x)}\end{aligned}$$

so as $x \to -\infty$

$$\lim_{x\to-\infty} f(x) = -\frac{\sqrt{2}}{1} = -\sqrt{2}$$

Marking scheme:

- Correct simplification when $x > 0$ — 1 mark
- Correct positive limit — 1 mark
- Correct simplification when $x < 0$ — 1 mark
- Correct negative limit — 1 mark

3 marks (b) Determine the vertical asymptote(s) of the graph $y = f(x)$. For each vertical asymptote $x = a$, determine whether each of the one-sided limits "equals" $+\infty$ or $-\infty$ as x approaches a.

Solution:

- Since the numerator and denominator are continuous functions for all x (in their domains), the only possible asymptote occurs when the denominator is zero. Namely where $x = -1$.
- As $x \to -1$ the numerator goes to $\sqrt{2+3} = \sqrt{5}$.
- As $x \to -1^+$ the denominator goes to zero but is positive. Hence

$$\lim_{x \to -1^+} f(x) = +\infty$$

- As $x \to -1^-$ the denominator goes to zero but is negative. Hence

$$\lim_{x \to -1^-} f(x) = -\infty$$

Marking scheme:

- Deducing asymptote only when denominator is zero — 1 mark
- Correct left-limit — 1 mark
- Correct right-limit — 1 mark

5 marks 5. Show that the following equation has at least one solution:

$$\cos x - \tan x = 0$$

Solution: We use the intermediate value theorem.

- Let $f(x) = \cos x - \tan x$. The function $\cos x$ is continuous everywhere and $\tan x$ is continuous on it's domain so in particular it is continuous on $(-\pi/2, \pi/2)$.
- Compute $f(0) = \cos 0 - \tan 0 = 1 > 0$
- Compute $f(\pi/4) = \cos(\pi/4) - \tan(\pi/4) = 1/\sqrt{2} - 1 < 0$ since $\sqrt{2} > 1$
- By the IVT since $f(x)$ is continuous on $[0, \pi/4]$ and is positive at $x = 0$ and negative at $x = \pi/4$, there exists some $0 < c < \pi/4$ so that $f(c) = 0$.

Marking scheme:

- Explaining function continuous — 2 marks
- Finding negative value — 1 mark
- Finding two positive values — 1 mark
- Invoking IVT — 1 marks

5 marks 6. Let $g(x)$ be a function satisfying

$$3x \leq g(x) \leq x^3 - 3x + 4$$

for all x. Now let

$$f(x) = \begin{cases} \sqrt{3-x} & x \leq 2 \\ g(x) - c & x > 2 \end{cases}$$

where c is a constant. Find the value of c that makes $f(x)$ continuous at $x = 2$. Ensure that your answer is fully justified; unjustified answers will not recieve credit.

Solution: In order for f to be continuous at $x = 2$, the left and right limits must agree and be equal to the value of the function.

- At $x = 2$, $f(x) = f(2) = \sqrt{3-2} = \sqrt{1} = 1$.
- The left-limit is

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \sqrt{3-x} = 1$$

  which is equal to $f(2)$.
- For the right limit

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} g(x) - c$$

  To find this limit we will use the Squeeze Theorem on $g(x)$. We have that

$$3x \leq g(x) \leq x^3 - 3x + 4$$

  As $x \to 2^+$ we have

$$\lim_{x \to 2^+} 3x = \lim_{x \to 2^+} x^3 - 3x + 4 = 6$$

  Hence, by the squeeze theorem

$$\lim_{x \to 2^+} g(x) = 6$$

- In order to make f continuous we therefore need $1 = 6 - c$ so $c = 5$.

Marking scheme:

- Realising need left-right limits to agree — 1 marks.
- Easy limit — 1 mark
- Harder limit — 2 marks
- Correct value — 1 mark.