Midterm Test 2 Duration: 60 minutes

*This test has **6 questions** on **9 pages**, for a total of 44 points.*

- Read all the questions carefully before starting to work.
- Q1 and Q2 are short-answer questions; put your answer in the boxes provided.
- All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Signature: ____

Question:	1	2	3	4	5	6	Total
Points:	6	12	4	7	7	8	44
Score:							

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (i) speaking or communicating with other examination candidates, unless otherwise authorized;
   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (iii) purposely viewing the written papers of other examination candidates;
   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Short-Answer Questions. Questions 1 and 2 are short-answer questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box, unless otherwise specified. Show your work also, for part marks. Each part of Q1 is worth 2 marks and each part of Q2 is worth 3 marks. Not all parts are of equal difficulty. **Simplify your answers as much as possible in Questions 1 and 2.**

2 marks

1. (a) Let $f(x) = \ln\left(\frac{1}{x\sqrt{3x+1}}\right)$. Find $f'(x)$.

Answer: $f'(x) = -\frac{1}{x} - \frac{3}{2(3x+1)}$

Solution: First we use log rules to simply our function

$$f(x) = -\ln\left(x\sqrt{3x+1}\right) = -\ln x - \frac{1}{2}\ln(3x+1)$$
$$f'(x) = -\frac{1}{x} - \frac{1}{2}\frac{1}{3x+1}\cdot 3$$

2 marks

(b) Evaluate $\arcsin\left(\sin\left(\frac{7\pi}{6}\right)\right)$.

Answer: $-\pi/6$

Solution: There are several ways to do this. First, $\sin(7\pi/6) = -\sin(\pi/6)$ observing the unit circle or the symmetry of the sine function. Now $\sin(\pi/6) = 1/2$ using either special triangles or the unit circle. To compute $\arcsin(-1/2)$ note that $\sin(-\pi/6) = -1/2$ and that $-\pi/6 \in [-\pi/2, \pi/2]$. Hence, $\arcsin(-1/2) = -\pi/6$.
Alternatively, note that $\sin(7\pi/6) = -\sin(\pi/6) = \sin(-\pi/6)$ using the unit circle and the oddness of sine. Then we have $\arcsin(\sin(-\pi/6)) = -\pi/6$ since $-\pi/6 \in [-\pi/2, \pi/2]$ which is the domain on which sine is invertible.

2 marks

(c) Find the derivative of $(\sin x)^{\ln x}$. You can assume $x \in (0, \pi)$.

Answer:
$(\sin x)^{\ln x}\left(\frac{1}{x}\ln(\sin x) + \ln x\frac{\cos x}{\sin x}\right)$

Solution: We use logarithmic differentiation. Let $y = (\sin x)^{\ln x}$ and take the log of both sides to see

$$\ln y = \ln x \ln(\sin x).$$

Now differentiate implicitly

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(\sin x) + \ln x\frac{1}{\sin x}\cos x.$$

Therefore,

$$\frac{dy}{dx} = (\sin x)^{\ln x}\left(\frac{1}{x}\ln(\sin x) + \ln x\frac{\cos x}{\sin x}\right).$$

3 marks 2. (a) Use a suitable linear approximation to estimate $\sqrt[3]{25}$.

Answer: $3 - \frac{2}{27}$

Solution: Let $a = 27$ and $f(x) = x^{1/3}$. We find

$$L(x) = f(a) + f'(a)(x - a).$$

Now $f(27) = 3$; $f'(x) = \frac{1}{3}x^{-2/3}$; $f'(a) = \frac{1}{3}(3)^{-2} = 1/27$ so

$$L(x) = 3 + \frac{1}{27}(x - 27).$$

Using the approximation $f(x) \approx L(x)$ we get

$$f(25) \approx L(25) = 3 + \frac{1}{27}(25 - 27) = 3 - \frac{2}{27}.$$

3 marks (b) **[For this problem provide brief justification for your choice.]**
Under the right conditions the velocity of an object with air resistance can be described by the following differential equation:

$$\frac{dv}{dt} = v^2$$

Which of the following functions satisfies the above equation?

A: $v(t) = 2t$

B: $v(t) = \frac{1}{3}t^3$

C: $v(t) = \frac{1}{1-t}$

D: $v(t) = e^{2t}$

E: None of the above

Answer: C

Solution: Observe that for $v(t) = \frac{1}{1-t} = (1-t)^{-1}$ we have

$$\frac{dv}{dt} = -1(1-t)^{-2}(-1) = \frac{1}{(1-t)^2} = v^2$$

and so the above differential equation is satisfied.

$\boxed{\text{3 marks}}$ (c) Find the equation of the line tangent to the curve $e^{2y} - (y-1)x = 4$ at the point $(3, 0)$.

Answer: $y = x - 3$

Solution: We differentiate implicitly with respect to x:

$$\begin{aligned} 2e^{2y}\frac{dy}{dx} - \frac{dy}{dx}x - (y-1) &= 0 \\ \frac{dy}{dx}\left(2e^{2y} - x\right) &= y - 1 \\ \frac{dy}{dx} &= \frac{y-1}{(2e^{2y} - x)} \end{aligned}$$

So at $(x, y) = (3, 0)$ we have a slope of $m = -1/(2-3) = 1$. Hence the equation of the tangent line is

$$y = x - 3$$

$\boxed{\text{3 marks}}$ (d) Suppose that a particle travels according to the equation $x\cos y + y\ln x = 0$. If its x-coordinate is changing at a constant rate of $2/3$ units per second what is the rate of change of its y-coordinate when the particle is at $(1, \pi/2)$.

Answer: $\pi/3$ units per second

Solution: We have $dx/dt = 2/3$ and need dy/dt when $(x, y) = (1, \pi/2)$. Differentiate both sides of the equation with respect to time:

$$\begin{aligned} \frac{dx}{dt}\cos y - x\sin y\frac{dy}{dt} + \frac{dy}{dt}\ln x + y\frac{1}{x}\frac{dx}{dt} &= 0 \\ \frac{dy}{dt}(\ln x - x\sin y) &= -\frac{dx}{dt}\left(\cos y + \frac{y}{x}\right) \end{aligned}$$

Substituting dx/dt and $x = 1$ and $y = \pi/2$ gives

$$\begin{aligned} -\frac{dy}{dt} &= -\frac{2}{3}(0 + \pi/2) \\ \frac{dy}{dt} &= \pi/3 \end{aligned}$$

Full-Solution Problems. In questions 3–6, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions**.

1 mark 3. (a) Use quotient rule to find the derivative of $\cot x$. Note that $\cot x = \frac{\cos x}{\sin x}$.

Solution:

$$\frac{d}{dx}\frac{\cos x}{\sin x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

3 marks (b) Show that

$$\frac{d}{dx}\Big(\text{arccot}(x)\Big) = -\frac{1}{1+x^2}.$$

Solution: Let $y = \text{arccot} x$. That is $\cot y = x$. Now differentiate with respect to x:

$$-\csc^2 y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = -\frac{1}{\csc^2 y}.$$

Recall that $\sin^2 x + \cos^2 x = 1$. Dividing this equation by $\sin^2 x$ gives the identity $1 + \cot^2 x = \csc^2 x$ and so we get

$$\frac{dy}{dx} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2}$$

the desired result, after noting the definition of y.

$\boxed{\text{5 marks}}$ 4. (a) A glass of iced tea is taken from the refrigerator and placed on the counter where it is forgotten about. After an hour has passed it is $12°C$. After two hours it is $17°C$. The temperature of the air around the counter is $25°C$. How cold is the refrigerator? You may assume the glass is warming according to Newton's law of cooling. Be sure to define all variables and units you use.

Solution: Let t be the time from the removal of the glass from the fridge in hours. Let $T(t)$ be the temperature of the glass at time t in Celsius. Call the ambient temperature $T_s = 25$. The equation for Newton's Law of Cooling is

$$T(t) - T_s = (T(0) - T_s)e^{kt}.$$

We have $T(1) = 12$ and $T(2) = 17$. We wish to find $T(0)$. We have two equations with two unknowns:

$$\begin{aligned} 12 - 25 = -13 &= (T(0) - 25)e^k \\ 17 - 25 = -8 &= (T(0) - 25)e^{2k} \end{aligned}$$

Dividing the two equations yields

$$e^k = \frac{8}{13} \Rightarrow k = \ln\left(\frac{8}{13}\right) < 0$$

Putting this back to the first equation gives

$$-13 = (T(0) - 25)\frac{8}{13} \Rightarrow T(0) = 25 - \frac{13^2}{8}$$

(which by the way is $3.875°C$).

$\boxed{\text{2 marks}}$ (b) If the glass is left on the counter indefinitely explain, using your model, what will happen to its temperature.

Solution: We have

$$T(t) = 25 + (T(0) - 25)e^{kt}$$

with $k = \ln\left(\frac{8}{13}\right)$. Since $\frac{8}{13} < 1$ we have that $k < 0$. In this way as $t \to \infty$ we see that $e^k t \to 0$ and so

$$\lim_{t\to\infty} T(t) = 25 + (T(0) - 25) \cdot 0 = 25.$$

Therefore, as time continues the glass will approach $25°C$, as your physical intuition would suspect.

3 marks 5. (a) Find the second degree Taylor Polynomial for $f(x) = x^{1/4}$ around $x = 1$.

Solution: Let $f(x) = x^{1/4}$ and $a = 1$. We compute

$$f'(x) = \frac{1}{4}x^{-3/4}, \quad f'(1) = \frac{1}{4}$$
$$f''(x) = -\frac{3}{16}x^{-7/4}, \quad f''(1) = -\frac{3}{16}$$

So using the formula for Taylor Polynomials we get

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$
$$T_2(x) = 1 + \frac{1}{4}(x-1) - \frac{3}{32}(x-1)^2$$

3 marks (b) Your friend uses your polynomial to approximate $2^{1/4}$. Find a bound on the error in this approximation.

Solution: We use

$$|R_2(2)| \leq \frac{M}{3!}|2-1|^3$$

where

$$|f'''(x)| \leq M$$

for $x \in [1, 2]$. Observe that

$$f'''(x) = \frac{3 \cdot 7}{16 \cdot 4}x^{-11/4}.$$

The x value between 1 and 2 which makes f''' the largest is $x = 1$. So

$$|f'''(x)| \leq \frac{3 \cdot 7}{16 \cdot 4}.$$

Therefore

$$|R_2(2)| \leq \frac{3 \cdot 7}{16 \cdot 4}\frac{1}{3 \cdot 2} = \frac{7}{64 \cdot 2} = \frac{7}{128}$$

1 mark (c) Show that the error is less than 0.1.

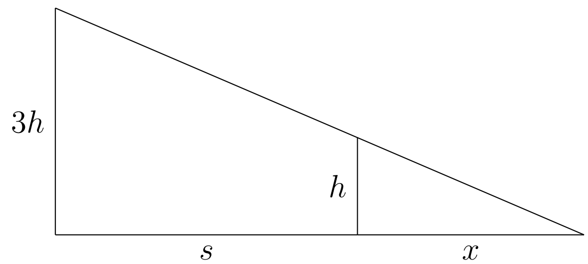
Solution: Note that $0.1 = 1/10 = 10/100$. Since $7 < 10$ and $128 > 100$ we have

$$|R_2(2)| \leq \frac{7}{128} < \frac{10}{100} = \frac{1}{10} = 0.1$$

Hence, the error in the approximation is less than 0.1 as desired.

4 marks 6. (a) A woman walks in a straight line away from a light post that is three times as tall as she is. Her distance from the post is given by $s(t) = 4t - \frac{1}{2}t^2$ where s is in meters and t is in seconds. Find the rate of change of the length of her shadow after 3 seconds. Be sure to define all variables you use.

Solution: We start by drawing and labelling a diagram. Let h be the height of the woman, x be the length of her shadow and s be her distance from the post.



Her speed is given by $v(t) = ds/dt = 4 - t$ and we require dx/dt. Using similar triangles we see

$$\begin{aligned} \frac{x}{h} &= \frac{s+x}{3h} \\ 3x &= s + x \\ x &= \frac{1}{2}s \end{aligned}$$

and so

$$\frac{dx}{dt} = \frac{1}{2}\frac{ds}{dt}.$$

After 3 seconds we have $v(3) = 4 - 3 = 1$ m/s so $dx/dt = 1/2$ m/s.

4 marks (b) What is the rate of change of the length of her shadow after she has walked a total of 10 meters?

Solution: Notice that at 4 seconds we have $v(4) = 4 - 4 = 0$. At this point she turns around and walks back toward the light post. After 4 seconds she has gone a total of $s(4) = 4 \cdot 4 - 4^2/2 = 8$ meters. So we want to know the time when she has walked an additional 2 meters, so when her distance from the post is 6 meters. Solve now

$$\begin{aligned} 6 = s(t) &= 4t - \frac{1}{2}t^2 \\ \frac{1}{2}t^2 - 4t + 6 &= 0 \\ \frac{1}{2}\left(t^2 - 8t + 12\right) &= 0 \\ \frac{1}{2}(t-6)(t-2) &= 0 \end{aligned}$$

So she is 6 meters from the post after 2 and 6 seconds. We choose 6 seconds because this time corresponds to her walking back toward the post. So at $t = 6$ we have $v(6) =$

$4 - 6 = -2$ meters per second. Therefore the rate of change of the length of her shadow is

$$\frac{dx}{dt} = \frac{1}{2}\frac{ds}{dt} = \frac{1}{2}(-2) = -1$$

in the units of meters per second. In other words the length of her shadow is shrinking at a rate of 1 m/s.