§ 3.8 Exponential Growth and Decay.

The simplest differential equation is

$$\frac{dy}{dt} = ky \quad \ldots \text{(1)}.$$

The goal is to find a function $y(t)$ satisfying (1).

k is a constant. If $k = 1$.

$$\frac{dy}{dt} = y.$$

Well $y(t) = e^t$ works.

In fact $y(t) = Ce^t$ works for any constant C.

For all of (1), k any constant.

$$y(t) = Ce^{kt}$$

Let's Check
$$\begin{aligned} y'(t) &= Ce^{kt} \cdot k \\ &= k \cdot Ce^{kt}. \\ &= k\,y(t) \end{aligned}$$

So, $y(t) = Ce^{kt}$ satisfies

$$\frac{dy}{dt} = ky.$$

What is C? Notice

$$y(0) = Ce^{k\cdot 0} = C.$$

This is y at time $t=0$. Called the initial condition.

$$\boxed{\begin{gathered} y(t) = y(0)\,e^{kt} \\ \text{solves} \\ \frac{dy}{dt} = ky \end{gathered}}$$

It turns out that this is the only solution to (1).

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Another well know differential equation is :

$$y'' = -y \quad \text{(Simple harmonic motion)}$$

$y(t) = \cos t$
$y'(t) = -\sin t$
$y''(t) = -\cos t$.

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This equation has two solutions.

A general differential equation may have many solutions.
They can be hard to find.

$$y'' + 2y' + y = 0.$$

Solutions are $y(t) = e^{-t}$
$y(t) = t e^{-t}$.

This can be verified by taking $y'(t)$ and $y''(t)$ and substituting.

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Population Growth

Let $P(t)$ be the population (bacteria).

$$\frac{dP}{dt} = kP$$

The constant k is the relative growth rate

$$k = \frac{1}{P}\frac{dP}{dt}.$$

We have $P(t) = P(0)e^{kt}$.

Example: A cell divides every 20 min. Initially you have 60 cells. Find the relative growth rate k?

Clicker Q: A: 20 → D: $\ln(2)/20$
B: $\ln(20)$
C: $\ln(2)$ E: $1/20$.

$$120 = P(20) = 60e^{k\cdot 20}$$

$$2\cancel{P(0)} = P(20) = \cancel{P(0)}e^{k\cdot 20}$$

$$2 = e^{k\cdot 20}$$

$$\ln(2) = k\cdot 20 \longrightarrow k = \frac{\ln(2)}{20}.$$

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240 min.

$\shortparallel$

Find the population after 4 hours.

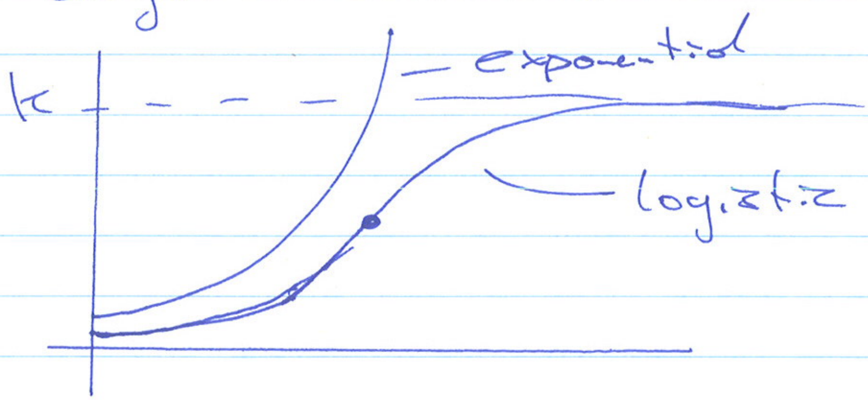
$$P(t) = P(0) \bullet e^{\frac{\ln 2}{20} t}$$

$$P(240) = 60\, e^{\frac{\ln 2}{20} \cdot 240} = 245{,}760$$

You can imagine after a few days the number of cells will be unrealistically large.

(There are more realistic models.)

Logistic Growth



Radiactive Decay: $m(t)$ is mass

$$\frac{dm}{dt} = km \quad (k<0).$$

The same as population growth just now k is negative and istead of doubling time we have half-life.

Newton's Law of Cooling.

The rate of change in temperature of an object is proportional to the difference between its temperature and that of its surroundings.

$$\frac{dT}{dt} = k(T - T_s)$$

T - temperature of object
t - time
k - constant
T_s - temperature of surroundings (fixed)

If we let $y = T - T_s$ then $\frac{dy}{dt} = \frac{dT}{dt}$.

So

$$\frac{dy}{dt} = ky .$$

⑦

$$y = T - T_S.$$

So, $y(t) = y(0)e^{kt}$.

we get $T(t) - T_S = (T(0) - T_S)e^{kt}$.

Example: My tea is 100°C.
After 2 minutes it is 93°C.
My office is 20°C.
When will the tea be 60°C?

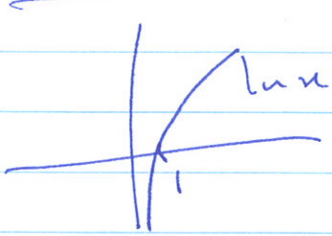
A: working
B: getting somewhere
C: stuck.

At $t=2$: $T(2) - T_S = (T(0) - T_S)e^{k \cdot 2}$.

$$93 - 20 = (100 - 20)e^{2k}.$$

$$\frac{73}{80} = \frac{80e^{2k}}{80}$$

$$\ln\left(\frac{73}{80}\right) = 2k. \Rightarrow k = \frac{1}{2}\ln\left(\frac{73}{80}\right).$$



$$k < 0.$$

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④ $T(t) - T_s = (T(0) - T_s)e^{kt}$.

Find t^* when $T(t^*) = 60$.

$$60 - 20 = (100 - 20)e^{kt^*}.$$

$$40 = 80e^{kt^*}$$

$$1/2 = e^{kt^*}$$

$$\ln(1/2) = kt^*$$

$$t^* = \frac{\ln(1/2)}{k}$$

$$= \frac{\ln(1/2)}{1/2 \ln\left(\frac{73}{80}\right)} > 0$$

$$\approxeq 15.1 \text{ minutes.}$$

What is the fate of my tea?

$$\lim_{t \to \infty} T(t)\ ?$$

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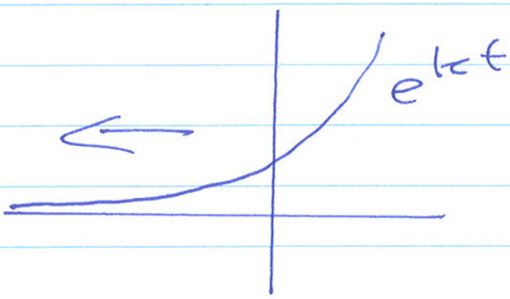
$$T(t) = T_s + (T(0) - T_s)e^{kt}.$$

$$\lim_{t \to \infty} T(t) = T_s + \lim_{t \to \infty} (T(0) - T(s))e^{kt}.$$



Note $k < 0$.

So $\lim_{t \to \infty} T(t) = T_s + 0$

$= T_s$.

Example: A body is found at 1:30 pm with a temp. of 32.5°C.
At at 2:30 pm the temp. is 30.3°C.
The room is 20°C.
When did the murder take place?
(Normal body temp. is 37°C).

$$T(t) = T_s + (T(0) - T_s)e^{kt}.$$

At 1:30pm (at $t=0$). $T(0) = 32.5$
At 2:30pm: $T(1) = 30.3$

~~32.5 = 20 = (~~

At $t=1$:

$$T(1) = T_s + (T(0) - T_s)e^k$$

$$30.3 - 20 = (32.5 - 20)e^k$$

(hours) $10.3 = 12.5e^k$

$$k = \ln\left(\frac{10.3}{12.5}\right) < 0.$$

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Find time when $T(t^*) = 37°C$.

$$T(t^*) - 20 = (32.5 - 20)e^{kt^*}$$

$$37 - 20 = (32.5 - 20)e^{kt^*}.$$

$$17 = 12.5\, e^{kt^*}$$

$$\frac{17}{12.5} = e^{kt^*}$$

$$kt^* = \ln(17/12.5)$$

$$t^* = \ln(17/12.5)/k$$

$$t^* = \frac{\ln(17/12.5)}{\ln\left(\frac{10.3}{12.5}\right)} < 0$$

$t^* \simeq -1.588$ hours $\simeq -95$ min.

So the murder took place at 11:55 am.