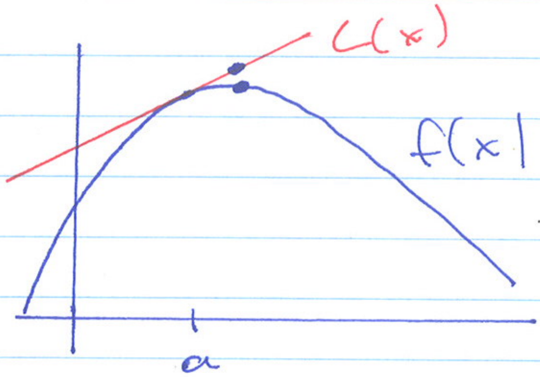
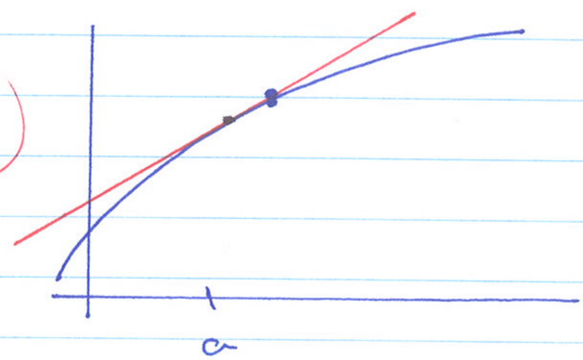
① Feb. 26

More Related Rates Notes Posted.

§ 3.10 Linear Approximation

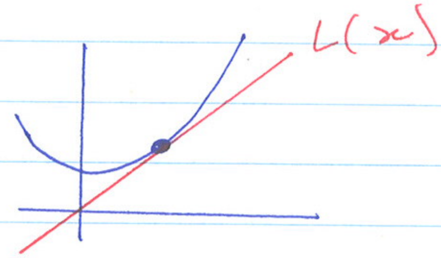
Clicker Q: Which linear approximation at $x = a$ will perform better?

A: 

B: 

Clicker Q: In A will the approximation $L(x) \approx f(x)$ be an over or under estimate?

→ A: over
B: under.

An example of under: 

Example: Approximate $\ln(0.9)$ via Linear Approximation.

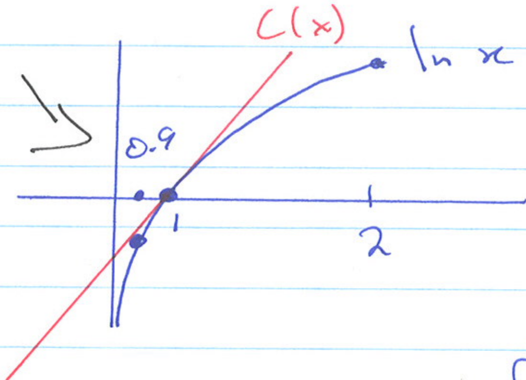
Note $\ln(1) = 0$. Need

$$L(x) = f(a) + f'(a)(x-a).$$

$$a = 1$$
$$f(x) = \ln x.$$

Idea: $L(x) \simeq f(x)$.

Over Estimate



$$f(a) = \ln(1) = 0$$
$$f'(x) = 1/x$$
$$f'(a) = 1/1 = 1.$$

So,
$$L(x) = 0 + 1(x-1)$$
$$= x - 1.$$

$$\ln(0.9) = f(0.9) \simeq L(0.9) = 0.9 - 1$$
$$= -0.1.$$

True value: $\ln(0.9) = -0.10536\ldots$

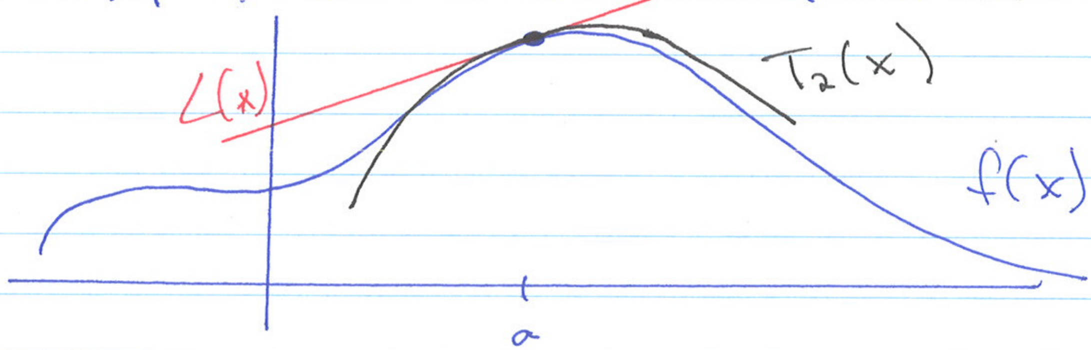
Example: An important example in physics is

$$\sin x \simeq x .$$

ie. $L(x) = x$ around $a = 0$.
Applications in optics, astronomy, gunnery.

(Course Notes §7) Taylor Polynomials.

We have just used linear functions to approximate a complicated function



Now $L(x)$ does a "good" job close to $x=a$ but what if we want to do better (get farther from $x=a$).

A related question: How does your calculator know the values of $\sin x$, $\cos x$, $\ln x$.

$T_2(x)$

Maybe we can find a quadratic polynomial to approximate $f(x)$.

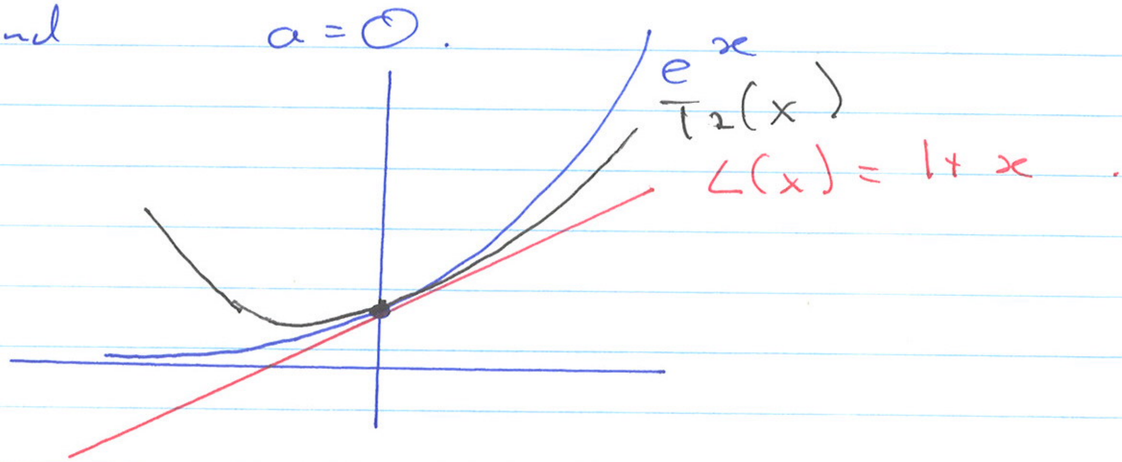
We will choose $T_2(x)$ so that it has the same second derivative as $f(x)$ at $x = a$.

ie. $T_2''(a) = f''(a)$.

$L(x)$: matches first derivative with $f(x)$
$T_2(x)$: matches both first and second derivative.

Example: Take $f(x) = e^x$.

We can show $L(x) = 1 + x$. around $a = 0$.



Find $T_2(x) = 1 + x + cx^2$.
So that

$$T_2''(0) = f''(0)$$

$$T_2''(x) = 2c \quad , \quad T_2''(0) = 2c$$

$$f''(x) = e^x \quad , \quad f''(0) = 1.$$

$$\Rightarrow \quad 2c = 1$$
$$c = 1/2.$$

So, $T_2(x) = 1 + x + \frac{1}{2}x^2$

- Second order approximation
- Taylor Polynomial of degree 2.

Taylor Polynomials around $a = 0$ are also called Maclaurin Polynomials.

⑥

Approximate $f(x) = e^x$ with $(a=0)$.

$$T_n(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots + c_n x^n.$$

and find $c_0, c_1, c_2, \dots, c_n$.

$c_0 = 1$

$c_1 = 1$ Set $f'''(0) = T_n'''(0)$

$c_2 = 1/2$

$c_3 = ?$

$$T_n'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \dots + n c_n x^{n-1}$$

$$T_n''(x) = 2c_2 + 3\cdot 2 c_3 x + \dots + n(n-1) c_n x^{n-2}.$$

$$T_n'''(x) = 3\cdot 2 c_3 + \dots + \cancel{n(n-1)(n-2) c_n x^{n-3}} + n(n-1)(n-2) c_n x^{n-3}.$$

$f'''(x) = e^x$, $f'''(0) = 1$.

$$T_n'''(0) = 3\cdot 2 c_3 + 0 = 3\cdot 2 c_3.$$

$$\Rightarrow \quad c_3 = \frac{1}{3\cdot 2}.$$

If we keep going we get

$$C_4 = \frac{1}{4 \cdot 3 \cdot 2} = \frac{1}{4!}$$

In general $C_n = \dfrac{1}{n!}$.

So, putting everything together.

$$e^x \simeq T_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots + \frac{1}{n!}x^n.$$

All this messy calculation was to convince you of the following formula!

For a general function $f(x)$ we have the Maclauren series.

$$\boxed{T_n(x) = \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \dots + \frac{f^{(n)}(0)x^n}{n!}}.$$

when $a = 0$.

Example: Find the 5^{th} degree Maclauren polynomial for $f(x)=\sin x$. $(a=0)$.

$$\left.\begin{aligned} f(0) &= \sin(0) = 0 \\ f'(0) &= \cos(0) = 1 \\ f''(0) &= -\sin(0) = 0 \\ f'''(0) &= -\cos(0) = -1 \end{aligned}\right\} \text{Pattern!}$$

$$f^{(4)}(0) = \sin(0) = 0$$

$$f^{(5)}(0) = \cos(0) = 1.$$

$$T_5(x) = 0 + 1\cdot x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!}$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Lets approximate $\sin(1)$ (1 rad.)

$$\sin(1) \simeq T_5(1) = 1 - \frac{1}{3!} + \frac{1}{5!}$$

$$= 0.8416\ldots$$

True Value: $\sin(1) = 0.84147\ldots$

You can also find Taylor Series away from 0 ($a \neq 0$).
The formula is,

$$\boxed{T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Example: Find $T_1(x)$, $T_2(x)$, $T_3(x)$ for
$f(x) = \ln x$ around $a = 1$.

Try to approximate $\ln(1/2)$.

$$T_1(x) = x - 1.$$
$$T_2(x) = x - 1 - \frac{1}{2}(x-1)^2.$$
$$T_3(x) = x - 1 - \frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3.$$

$\ln(1/2) \simeq T_1(1/2) = -0.5$
$\ln(1/2) \simeq T_2(1/2) = -0.625$
$\ln(1/2) \simeq T_3(1/2) = -0.\overline{6}$

True Value: $\ln(1/2) = -0.693\ldots$
More terms will get more decimal places.