(1) Feb. 5

Midterm (with Solutions) Posted.

Find the derivative of

1. $\sin x \cdot g(x) \Rightarrow \cos x\, g(x) + \sin x\, g'(x)$
2. $[g(x)]^2 \Rightarrow 2g(x) \cdot g'(x)$.

§ 3.5 Implicit Differentiation.

Clickers:

A: Ok
B: Lost/Confused/Question
C: Too Easy.

Now differentiate implicitly, think of y as a function of x.

3. $\sin x \cdot y = 1$.

$$\frac{d}{dx}(\sin x\, y) = \frac{d}{dx}(1) = 0$$

$$\cos x\, y + \sin x\, y' = 0$$. (Solve for y').

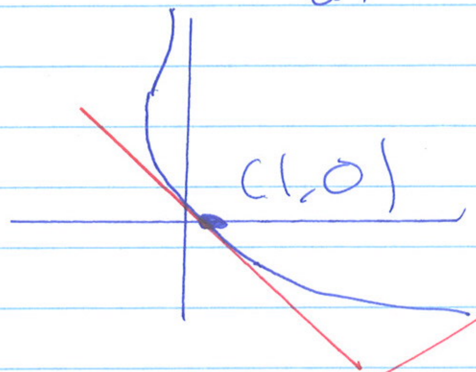
4. $x^3 + y^3 + y^2 + xy = 2$.

Take $\frac{d}{dx}$ of both sides

$$3x^2 + 3y^2y' + 2yy' + xy' + y = 0$$

1 Feb. 5

Example: Find the equation of the line tangent to $xe^y + y = 1$ at the point $(1, 0)$.



$$\frac{d}{dx}\left(xe^y + y\right) = \frac{d}{dx}(1)$$

Differentiate

$$e^y + e^y x \frac{dy}{dx} + \frac{dy}{dx} = 0.$$

Solve/Plug In:

$$e^y x \frac{dy}{dx} + \frac{dy}{dx} = -e^y$$

$$\frac{dy}{dx}\left(e^y x + 1\right) = -e^y$$

$$\frac{dy}{dx} = \frac{-e^y}{e^y x + 1}.$$

At $(1, 0)$

$$m = \frac{-e^0}{e^0 \cdot 1 + 1} = -1/2.$$

$$y - y_0 = m(x - x_0)$$

$$y - 0 = -\tfrac{1}{2}(x - 1)$$

$$y = -\tfrac{1}{2}(x - 1).$$

③ Feb. 5

§3.5

Derivatives of Inverse Trig. Functions.

Consider $y = \sin^{-1} x$.
Want to find dy/dx.

Recall that $y = \sin^{-1} x$ means $\sin y = x$
$-\pi/2 \le y \le \pi/2$.

Let's differentiate implicitly with respect to x.

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x).$$

$$\cos y \cdot y' = 1$$

$$y' = \frac{1}{\cos y}.$$

Last class we used $\cos^2 y + \sin^2 y = 1$ to simplify

$$\cos y = +\sqrt{1-\sin^2 y}.$$

Since $-\pi/2 \le y \le \pi/2$, $\cos y \ge 0$
So take 'positive' root.

4 Feb. 5

So, $y' = \dfrac{1}{\sqrt{1-\sin^2 y}}$.

$y' = \dfrac{1}{\sqrt{1-x^2}}$ //

Hence,

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}.$$

(Table p. 214).

Example: Find $\dfrac{d}{dx}\left(\tan^{-1} x\right)$.

(Hint: • $\dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}$ Divide by $\cos^2\theta$

• $\tan^2\theta + 1 = \sec^2\theta$)

$y = \tan^{-1} x \rightarrow (\tan y = x)$.

$\dfrac{d}{dx}$: $\sec^2 y \cdot \dfrac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}.$$

5 Feb. 5

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

If the problem doesn't have y's to start you should get rid of them in your final answer.

§ 3.6 Derivatives of Logarithmic Functions

So far we have avoided taking the derivative of $f(x) = \log_a x$.

This can be done with implicit differentiation.

Let $y = \log_a x$ that is

$$a^y = x.$$

Differentiate Implicitly

$$\frac{d}{dx}(a^y) = \frac{d}{dx}(x)$$

$$a^y \ln a \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}.$$

So, $$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}}.$$

In particular of $a = e$ we have
$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}.$$

Example: Using our new rule find the derivative of $f(x) = \ln(x^2+1)$

Chain Rule: $f'(x) = \dfrac{1}{x^2+1} \cdot 2x = \dfrac{2x}{x^2+1}$.

Example: What about $f(x) = \ln\left(\dfrac{x+1}{\sqrt{x-2}}\right)$.

$$f'(x) = \frac{1}{\frac{x+1}{\sqrt{x-2}}} \cdot \frac{1 \cdot \sqrt{x-2} - (x+1) \cdot \frac{1}{2}(x-2)^{-1/2}}{(x-2)}.$$

Chain/Quotient Rule

OR. We can simplify before taking the derivative.

$$f(x) = \ln(x+1) - \ln(\sqrt{x-2})$$
$$= \ln(x+1) - \ln\left((x-2)^{1/2}\right)$$
$$= \ln(x+1) - \frac{1}{2}\ln(x-2)$$
$$f'(x) = \frac{1}{x+1}\cdot(1) - \frac{1}{2}\,\frac{1}{x-2}\cdot 1$$
$$= \frac{1}{x+1} - \frac{1}{2(x-2)}.$$

Both methods are correct and will give the same answer.

We can use this logarithm simplification in other situations.

$$y = f(x) = \frac{(x+1)^2(2x+3)^4(3x+5)^6(4x+7)^{8}}{\sqrt{x^2+1}}.$$

We can do this using quadruple product rule (chain rule) inside quotient rule. OR

We can take ln and ~~differentiate~~ differentiate implicitly.

(8) Feb. 5

$$\ln y = \ln\left(\frac{(x+1)^2(2x+3)^4(3x+5)^6(4x+7)^8}{(x^2+1)^{1/2}}\right)$$

$$\ln y = 2\ln(x+1) + 4\ln(2x+3) + 6\ln(3x+5) + 8\ln(4x+7) - \frac{1}{2}\ln(x^2+1).$$

Take d/dx of both sides.

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2}{x+1}\cdot(1) + \frac{4}{2x+3}\cdot 2 + \frac{6}{3x+5}\cdot 3 + \frac{8}{4x+7}\cdot 4 - \frac{1}{2}\frac{1}{x^2+1}\cdot 2x.$$

Solve for dy/dx.

$$\frac{dy}{dx} = y\underbrace{\left(\frac{2}{x+1} + \frac{8}{2x+3} + \frac{18}{3x+5} + \frac{32}{4x+7} - \frac{x}{x^2+1}\right)}$$

$$= \frac{(x+1)^2(2x+3)^4(3x+5)^6(4x+7)^8}{\sqrt{x^2+1}}\left(\qquad\right)$$

⑧ Feb. 5.

Sometimes logarithmic differentiation is optional, sometimes it is necessary

Example: $f(x) = x^x$ — not a polynomial
not an exponential function.

Let $y = x^x$.

$\ln y = \ln(x^x) = x \ln x$

Take $\frac{d}{dx}$ of both sides:

$$\frac{1}{y} y' = \ln x + \frac{x}{x} = \ln x + 1.$$

Solve for y' and put back x^x:

$$y' = y(\ln x + 1)$$

$$f'(x) = y' = x^x(\ln x + 1)$$