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Midterm: Feb. 3 in class.

§ 3.1 Derivatives of Polynomials and Exponentials

Last class we used power rule to find the derivatives of polynomials.

What about $f(x) = 2^x$?

$$f'(x) \neq x\,2^{x-1}$$

not a polynomial
this is an exponential function.

Take $f(x) = a^x$ where $a > 0$.
Use the limit definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x (a^h - 1)}{h}.$$

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$$f'(x) = \underset{\text{original function.}}{a^x} \underbrace{\lim_{h\to 0} \frac{a^h - 1}{h}}_{\text{just a numbe}} .$$

For $f(x) = 2^x$: $f'(x) = 2^x \lim_{h\to 0} \frac{2^h - 1}{h}$

$$= 2^x (0.693\ldots)$$

For $f(x) = 3^x$: $f'(x) = 3^x \lim_{h\to 0} \frac{3^h - 1}{h}$

$$= 3^x (1.098\ldots)$$

It looks like there is a number between 2 and 3 where

$$f'(x) = f(x).$$

This number is $e = 2.71828\ldots$.
This is the number so that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1 .$$

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So, $(e^x)' = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x$.

This function comes up a lot when solving "differential equations"

These equations are what scientists and engineers use to model things.

Example: Find the equation of the line tangent to the curve $y = f(x) = -3e^x + x^{-1/2}$ at the point $x = 1$.

First, find $f'(x) = -3e^x - \frac{1}{2}x^{-3/2}$.

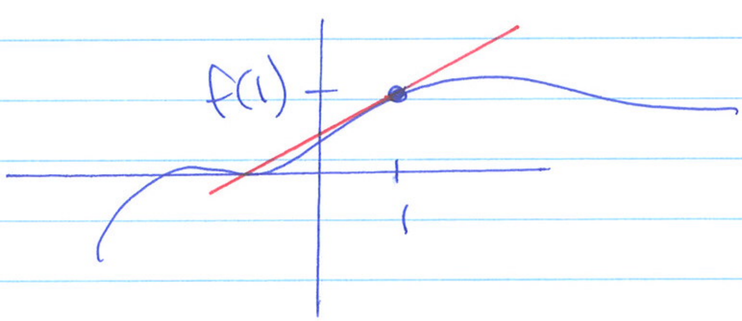
The slope at $x = 1$ is: $f'(1) = -3e - \frac{1}{2}$.

We also need a point $(x, y) = (1, f(1)) = (1, -3e + 1)$.

So, $y - y_1 = m(x - x_1)$

$y - (-3e + 1) = \left(-3e - \frac{1}{2}\right)(x - 1)$.

slope point form.



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aside: $y = mx + b$ — Solve.

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§ 3.2 The Product and Quotient Rules

Clicker Q: What is the derivative of

$$f(x) = (x^2)(x^3) = x^5.$$

A: $(2x)(3x^2)$ C: $5x^4$
B: $(2x)(x^3) + (x^2)(3x^2)$ D: $6x^3$.

Both B and C.

Product Rule (p. 184) If f and g are differentiable functions then.

$$\frac{d}{dx}(f \cdot g) = \frac{df}{dx} g + f \frac{dg}{dx}.$$

The proof comes from the limit definition.

Example: $f(x) = xe^x$

$(fg)' = f'g + g'f$.

Clickers:
A: working
B: getting somewhere
C: stuck.

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$$f'(x) = (xe^x)' = (x)'e^x + x(e^x)'$$
$$= e^x + xe^x$$
$$= (1+x)e^x.$$

Quotient Rule: If f and g are differentiable then
(p. 186)

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

"low d-high minus high d-low, square the bottom and away we go".

Example: Find $f'(x)$ where $f(x) = \dfrac{e^x + 4}{x^2+2x}$.

$$f'(x) = \frac{(e^x+4)'(x^2+2x) - (e^x+4)(x^2+2x)'}{(x^2+2x)^2}$$

$$= \frac{e^x(x^2+2x) - (e^x+4)(2x+2)}{(x^2+2x)^2}.$$

$$= \frac{x^2e^x + 2xe^x - 2xe^x - 2e^x - 8x - 8}{(x^2+2x)^2}$$

$$= \frac{x^2e^x - 2e^x - 8x - 8}{(x^2+2x)^2}.$$

not always necessary, but may be useful.

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Example: $f(x) = \dfrac{x^7 \cdot x^{-1} + x}{x^2}$.

Find $f'(x)$.

We could use product/quotient rule.

Or $f(x) = \dfrac{x^7 x^{-1}}{x^2} + \dfrac{x}{x^2}$

$= x^4 + x^{-1}$

So $f'(x) = 4x^3 - 1x^{-2}$.

Simplification can be useful before taking the derivative.

§ 3.3 Derivatives of Trig. Functions

Let's find the derivative of $f(x) = \sin x$.

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}.$$

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$$= \lim_{h\to 0} \frac{\overbrace{\sin(x)\cos(h) + \cos(x)\sin(h)}^{\text{trig. identity.}} - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h}$$

$$+ \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h}.$$

$$= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}.$$

But what are these limits?

With a lot of work and the squeeze theorem we can show

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0, \quad \lim_{h\to 0} \frac{\sin h}{h} = 1.$$

So, $f'(x) = \sin x \cdot 0 + \cos x \cdot 1$
$= \cos x.$

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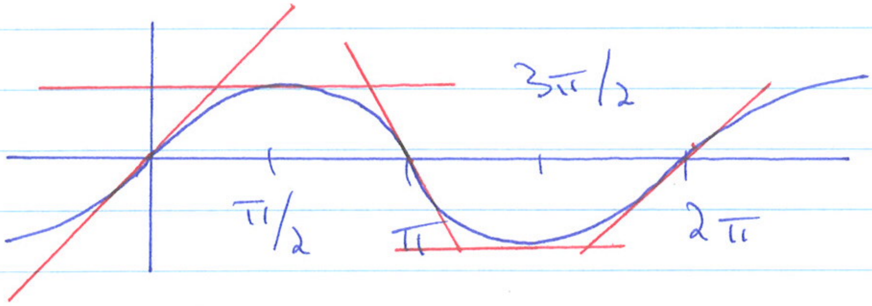

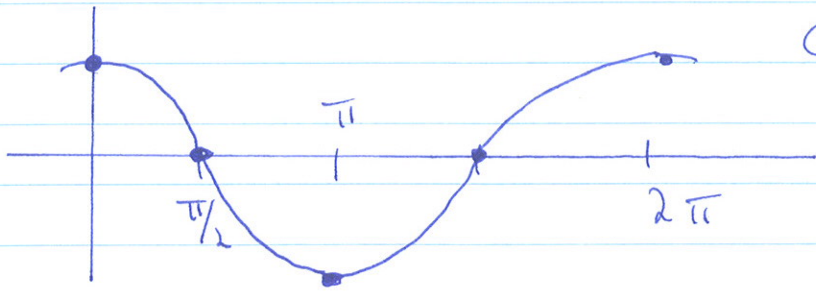

$$\left.\begin{array}{ll} \text{So,} & \dfrac{d}{dx}\sin x = \cos x \\ \text{similarly,} & \dfrac{d}{dx}\cos x = -\sin x \end{array}\right\}$$

With these we can find the derivatives of the other trig. functions.

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

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Similarly (p. 193)

$\csc x = \dfrac{1}{\sin x} \longrightarrow (\csc x)' = -\cot x \csc^2 x$

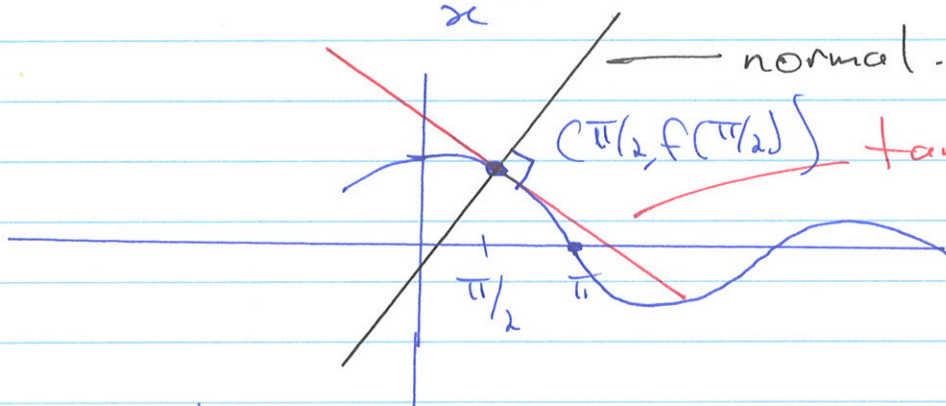
$\sec x = \dfrac{1}{\cos x} \longrightarrow (\sec x)' = \sec x \tan x$

$\cot x = \dfrac{1}{\tan x} \longrightarrow (\cot x)' = -\csc^2 x.$

quotient rule.

Example:

Find the equation of the normal line to $f(x) = \dfrac{\sin x}{x}$ at the point $x = \pi/2$.



First, $f'(x) = \dfrac{x\cos x - \sin x}{x^2}$.

$f'(\pi/2) = \dfrac{\pi/2 \cos(\pi/2) - \sin(\pi/2)}{(\pi/2)^2}$

$= -1/_{\pi^2/4} = -4/_{\pi^2}$.

slope of the tangent line.

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We want the line with slope
~~$m = 4/\pi^2$~~. $m = \pi^2/4$.

negative ~~reciprocal~~...
reciprocal...

The normal line is perpendicular, to the tangent line

We need the point

$$\left(\pi/2,\ f(\pi/2)\right) = \left(\pi/2,\ 2/\pi\right)$$

So,

$$y - \frac{2}{\pi} = \frac{\pi^2}{4}\left(x - \pi/2\right)$$

is the equation of the normal line.