① March 10.

Midterm: Thursday at 8am 60min.

Office Hours: Today: 2:30 - 4pm } LSK 300C
Wed: 9:30 - 11am }

Workshop TA Review Session:
Today at 6pm in MATX 1101.

Clickers: A: OK
B: Lost / Question
C: Bored.

§ 4.2 The Mean Value Theorem

You are driving along the highway.
(Speed limit 100 km/h)
You travel 240 km in 2 hours.
You were definitely speeding.
We don't know at what time but we know there was such a time.

Your average speed was $\frac{240 \text{ km}}{2 \text{ h}} = 120 \text{ km/h}$

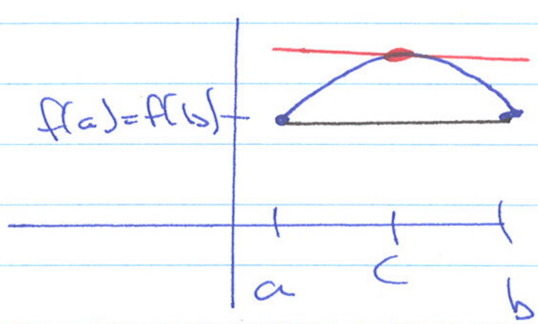
At some point your instantaneous speed must have been 120 km/h.

Idea: If you have a 'nice' function than somewhere the instantaneous rate of change is equal to the average rate of change.

Let's start small.

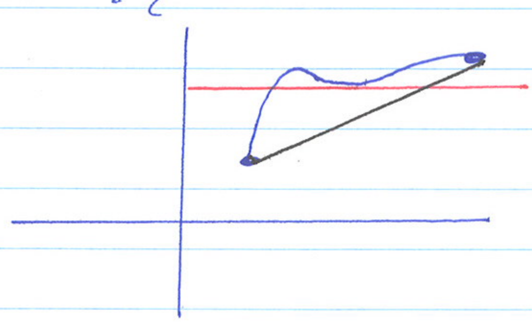
Rolle's Theorem: (baby MVT).

Let f be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $f(a) = f(b)$. Then there exists a c between a and b such that $f'(c) = 0$.

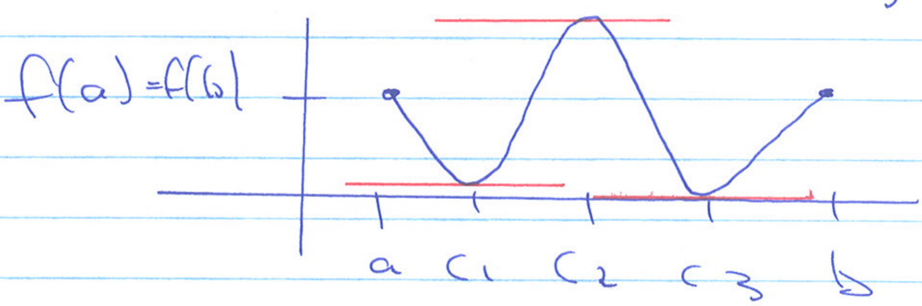


In other words if $f(x)$ is nice and touches the same value twice it has a horizontal tangent line.

asside:

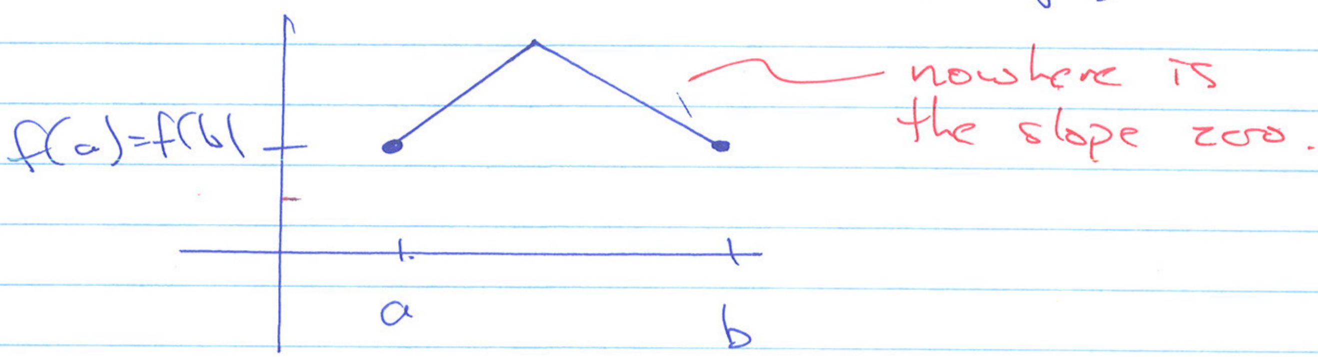


There could be many values.



We can't always find the c but we know there is at least one.

Ex: If f isn't 'nice' (not differentiable, say)



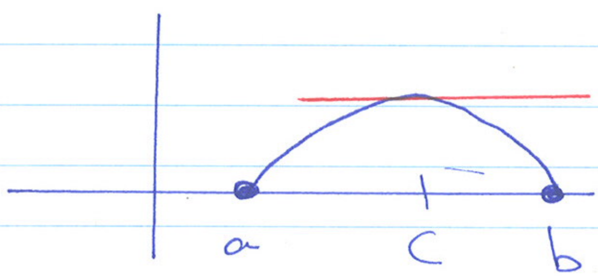
Rolle's Theorem is great for proving facts about the number of roots of a function.

Example: Prove that $f(x) = x^3 + x - 1$ has exactly one real root.

Idea: We can show it has at least one root using IVT.

The idea is to use proof by contradiction. We assume that $f(x)$ has two roots and try to get something to go wrong.

If it has two roots, ~~that~~ then it will look something like:



$f(a) = f(b) = 0$.
So somewhere $f'(c) = 0$.

But maybe we can show that $f'(x) \neq 0$ and so this picture is impossible.

Solution: First, we show there is at least one solution. Compute:

$$f(0) = -1, \quad f(1) = 1 + 1 - 1 = +1.$$

Since f is continuous by IVT it has a root.

Next suppose there are two distinct roots. So $f(a) = 0$ and $f(b) = 0$. Now $f(x)$ is continuous/differentiable so by Rolle's Theorem

$$f'(c) = 0$$

for some c between a and b.

However,

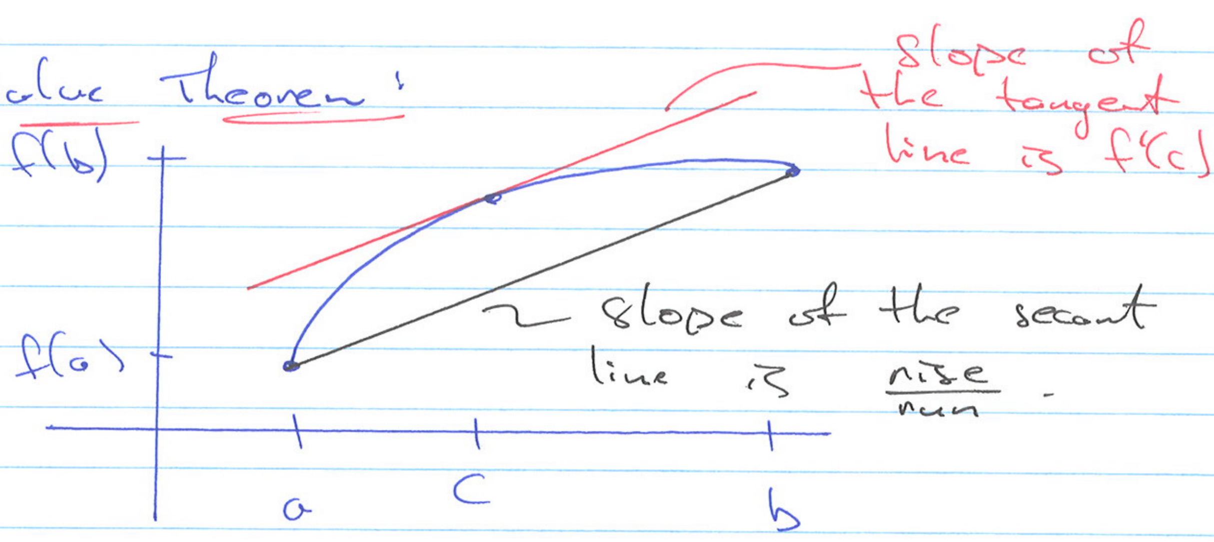
$$f(x) = x^3 + x - 1$$
$$f'(x) = 3x^2 + 1 > 0.$$

for all x.

We have reached a contradiction and so our assumption that we had two roots is false. Therefore $f(x)$ has exactly one root. $\boxtimes$

The MVT is like Rolle's Theorem but tilted.

Mean Value Theorem:



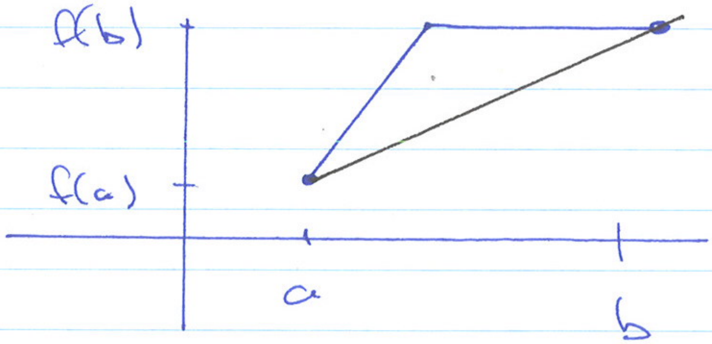
(6)

Suppose that f is continuous on [a,b] and differentiable on (a,b). Then there is a c between a and b where

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}.$$

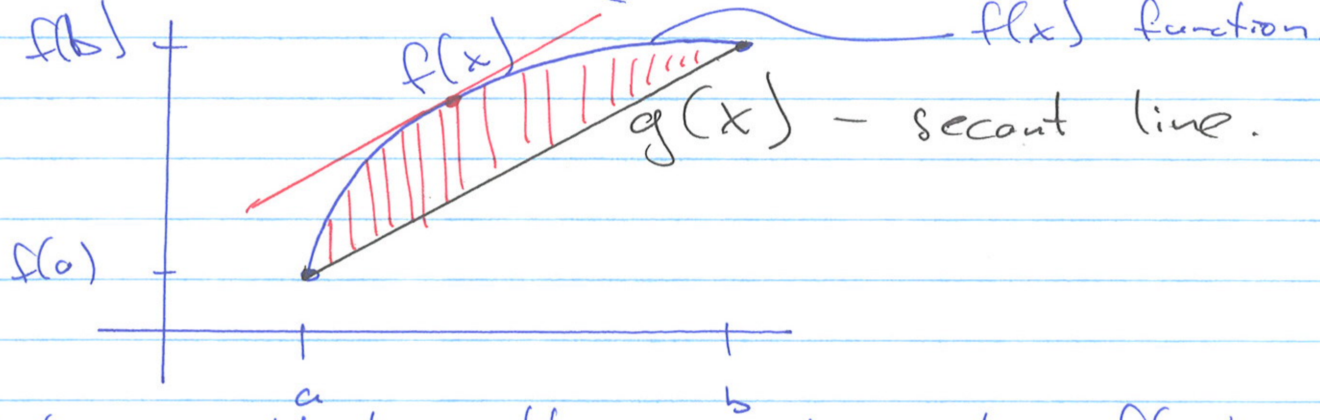
Somewhere the instantaneous speed is equal to the average speed.

Again if the function isn't nice the result might no be true.

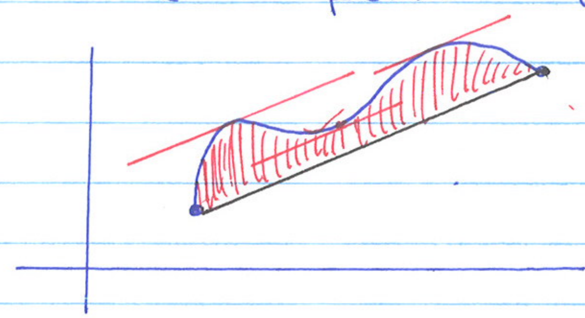


Proving Rolle's Theorem is hard but once we have RT proving MVT is ok.

Proof of MVT: (uses Rolle's Theorem)



Notice that the point when f(x) is farthest (or closest) to g(x) is the point we want.



Consider the difference,

$$h(x) = f(x) - g(x)$$

So,

$$h'(x) = f'(x) - g'(x).$$

Now $h(a) = f(a) - g(a) = 0$
$h(b) = f(b) - g(b) = 0$.

By Rolle's Theorem (since h is nice)

$h'(c) = 0$ for some $c \in (a,b)$.

So, $0 = h'(c) = f'(c) - g'(c)$.

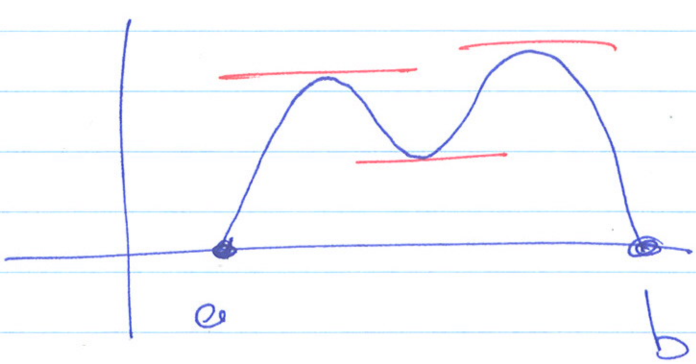
So, $f'(c) = g'(c)$.

What is $g'(x)$? Just $\frac{\text{rise}}{\text{run}}$.

$$g'(x) = \frac{f(b) - f(a)}{b - a}.$$

So, putting everything together,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad //$$



$h(x)$.

Q #35 in Section 4.2

Two runners start a race and finish in a tie. Prove that at some point during the race they had the same speed.

in time

Try this later (same as proof of MVT).

Example: Suppose $f(0) = -3$ and $f'(x) \le 5$ for all x. What is the largest $f(2)$ can be?

Consider the interval $[0, 2]$. $f(x)$ has a derivative so it is nice. By MVT there is a c with

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{f(2) - (-3)}{2}$$

$$f'(c) = \frac{f(2) + 3}{2}.$$

or $$2f'(c) = f(2) + 3$$
$$2f'(c) - 3 = f(2).$$

So, $$f(2) = 2f'(c) - 3$$
$$\leq 2 \cdot 5 - 3 = 7.$$

So, $$f(2) \leq 7.$$

Example: The MVT plays a part in Taylor Polynomials with remainder.

Let's use a zeroth order Taylor polynomial to approximate $\sqrt{99}$.

Let $f(x) = \sqrt{x}$. Let $a = 100$.

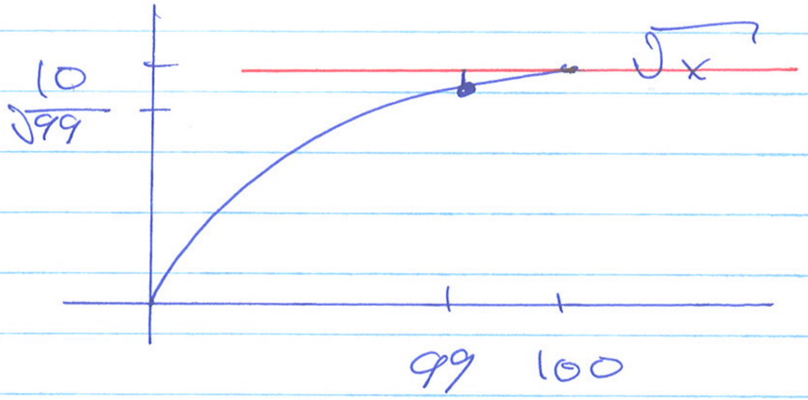
$$T_n(x) = f(a) + f'(a)(x-a) + \cdots .$$

$$T_0(x) = f(a)$$

So, $$T_0(x) = \sqrt{100} = 10.$$

So, $$f(x) \simeq T_0(x)$$
$$f(99) \simeq 10.$$

11



Let $f(x) = T_0(x) + R_0(x)$.

Let's estimate the remainder using MVT.

Consider $f(x)$ on $[99, 100]$.

f is nice so we can use MVT:

$$f'(c) = \frac{f(100) - f(99)}{100 - 99}$$

$$= \frac{\sqrt{100} - \sqrt{99}}{100 - 99}.$$

Now $R_0(99) = f(99) - T_0(99)$

$= \sqrt{99} - 10$.

(12)

$$f'(c) = \frac{-(\sqrt{99} - 10)}{100 - 99}$$

$$= \frac{-R_0(99)}{100-99}.$$

$$R_0(99) = -f'(c)(100-99).$$

The remainder formula is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

$$R_0(x) = \frac{f^{(0+1)}(c)}{1!}(x-a)^{0+1}.$$

$$= f'(c)(x-a).$$

$$R_0(99) = f'(c)(\underset{x}{\underset{\shortparallel}{99}} - \underset{a}{\underset{\shortparallel}{100}}).$$

and c is between x and a.

13.

So, let's finish the problem.

Find M with

$$|f'(c)| \leq M$$

for c between 99 and 100.

In this way $|R_0(99)| \leq M|99-100| = M$.

So, $f'(x) = \dfrac{1}{2\sqrt{x}}$ on $[99, 100]$.

$$|f'(c)| \leq \frac{1}{2\sqrt{99}} < \frac{1}{2\sqrt{81}} = \frac{1}{2 \cdot 9} = \underbrace{\frac{1}{18}}_{M}.$$

So, $|R_0(x)| \leq \dfrac{1}{18}$.

So

$$\underbrace{10 - \frac{1}{18}}_{9.94\ldots} < \underbrace{\sqrt{99}}_{9.95\ldots} < 10$$