① March 24

Go over Midterm: Wed: 5-7pm in LSK 201.

§ 4.5 Curve Sketching.

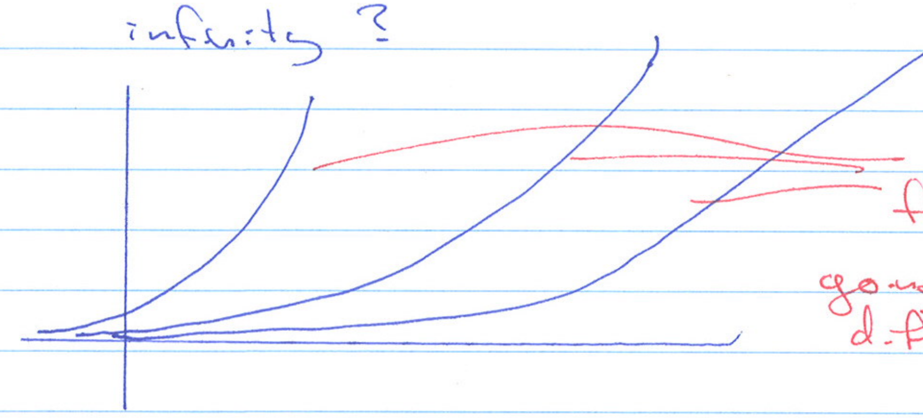
There's one more kind of behaviour we study in this course.

Example: $f(x) = \dfrac{x^3 + 2x^2}{x^2+1}$.

Horizontal asymptote?

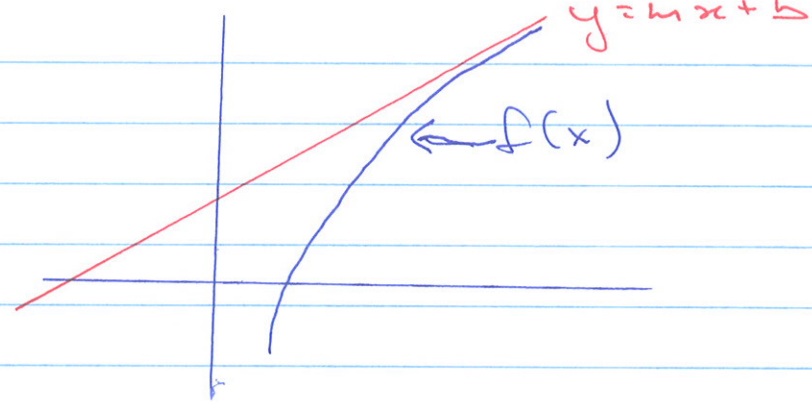
$$\lim_{x\to\infty} \frac{x^3+2x^2}{x^2+1} = \lim_{x\to\infty} \frac{x+2}{1+1/x^2} = \infty.$$

This function is going to infinity as $x \to \infty$. But how is it going to infinity?



all these functions are going to infinity differently.

Some functions approach a particular line as $x \to \pm\infty$. These functions have slant asymptotes.



$$\lim_{x \to \infty} f(x) - (mx+b) = 0.$$

The difference between f(x) and the line goes to zero as $x \to \infty$.

Example: $f(x) = \dfrac{x^3 + 2x^2}{x^2+1}$.

Consider the line $y = mx + b$. and take the limit:

$$\lim_{x \to \infty} f(x) - (mx+b)$$

$$= \lim_{x \to \infty} \frac{x^3+2x^2}{x^2+1} - (mx+b)$$

$$= \lim_{x \to \infty} \frac{x^3+2x^2}{x^2+1} \quad \frac{-(x^2+1)(mx+b)}{x^2+1}$$

$$= \lim_{x \to \infty} \frac{x^3+2x^2-(x^2+1)(mx+b)}{x^2+1}$$

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$$= \lim_{x \to \infty} \frac{x^3 + 2x^2 - mx^3 - bx^2 - mx - b}{x^2 + 1}$$

take $m = 1$. $b = 2$.

$$= \lim_{x \to \infty} \frac{\cancel{x^3} + \cancel{2x^2} - \cancel{x^3} - \cancel{2x^2} - x - 2}{x^2 + 1}$$

$$= \lim_{x \to \infty} \frac{-x - 2}{x^2 + 1} = \lim_{x \to \infty} \frac{-1/x - 2/x^2}{1 + 1/x^2}$$

$$= 0.$$

So, $y = x + 2$ is a slant asymptote.

or, $f(x)$ has a slant asymptote:

$y = x + 2$.

We also have the othe direction:

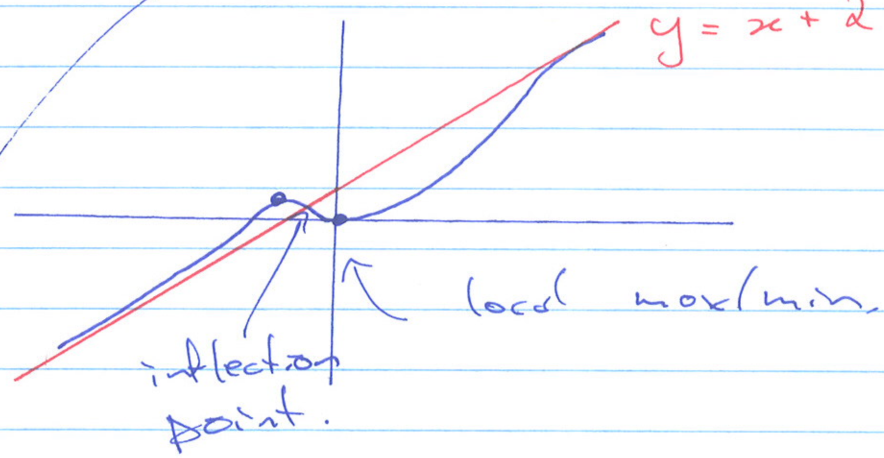
try $y = x + 2$ as $x \to -\infty$.

$$\lim_{x \to -\infty} f(x) - (x + 2) = \lim_{x \to -\infty} \frac{x^3 + 2x^2}{x^2 + 1} - \frac{(x+2)(x^2+1)}{x^2 + 1}.$$

$$= \lim_{x \to -\infty} \frac{\cancel{x^3} + \cancel{2x^2} - \cancel{x^3} - \cancel{2x^2} - x - 2}{x^2 + 1} = 0.$$

$\Rightarrow$ slant asymptote of $y = x + 2$ as $x \to -\infty$.

$$f(x) = \frac{x^3 + 2x^2}{x^2+1}$$ (With the rest of curve sketching)



You can also find $y = x+2$ by long division.

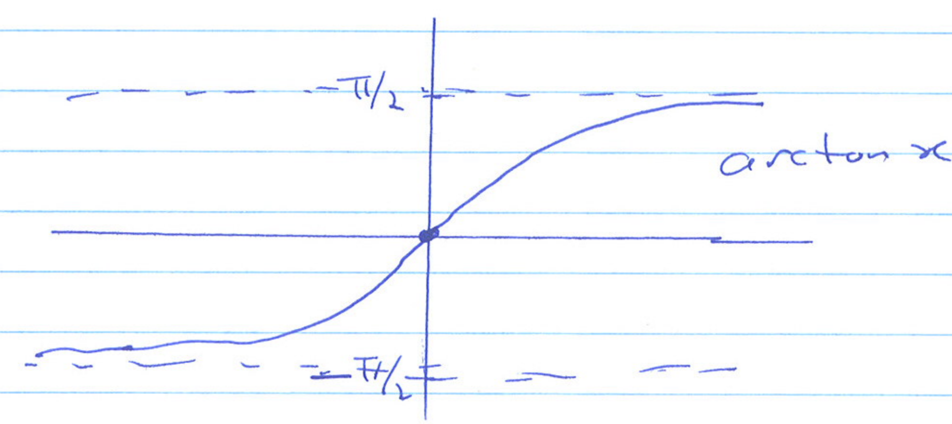
Example: Find S.A. and sketch.

$$f(x) = x - \arctan x.$$

$$\lim_{x \to \infty} x - \arctan x - (mx + b)$$

$$= \lim_{x \to \infty} x - \arctan x - mx - b.$$

take $m = 1$. $b = -\pi/2$.



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$$\lim_{x\to\infty} \cancel{x} - \arctan x - \cancel{x} + \pi/2.$$

$$= -\lim_{x\to\infty} +\arctan x + \pi/2.$$

$$= -\pi/2 + \pi/2 = 0.$$

So, slant asymptote of $y = x - \pi/2$.

We can also show that $y = x + \pi/2$ is also a slant asymptote $(x \to -\infty)$.
Let's show this.

$$\lim_{x\to-\infty} f(x) - \left(x + \pi/2\right)$$

$$= \lim_{x\to-\infty} \cancel{x} - \arctan x - \cancel{x} - \pi/2.$$

$$= \lim_{x\to-\infty} -\arctan x - \pi/2.$$

$$= \pi/2 - \pi/2 = 0.$$

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Let's sketch $f(x) = x - \arctan x$.

1. Domain: $\mathbb{R}$
2. Intercepts: $(0, 0)$.
3. Inc/Dec. Local Max/Min
4. Concavity Inflection points

3. $f'(x) = 1 - \dfrac{1}{1+x^2}$. (always exists)

$f'(x) = 0$ when: $x = 0$.

	$(-\infty, 0)$	$(0, \infty)$
$f(x)$	$\nearrow$	$\nearrow$
$f'(x)$	> 0.	> 0

No local max/min.

4. $f''(x) = (1+x^2)^{-2} \cdot 2x = \dfrac{2x}{(1+x^2)^2}$.

$f''(x) = 0$ when $x = 0$.

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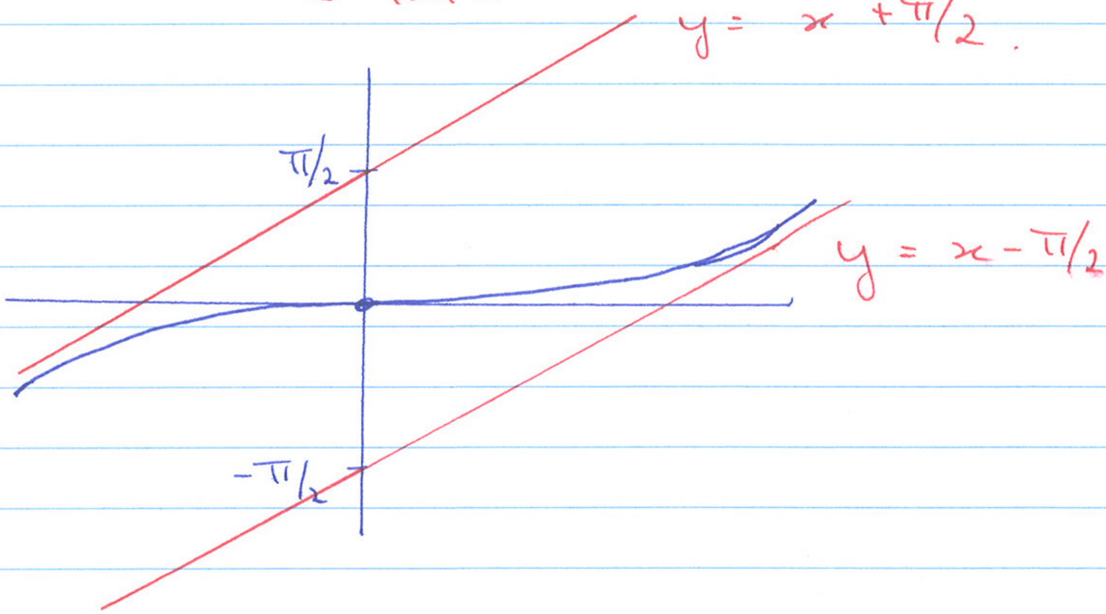
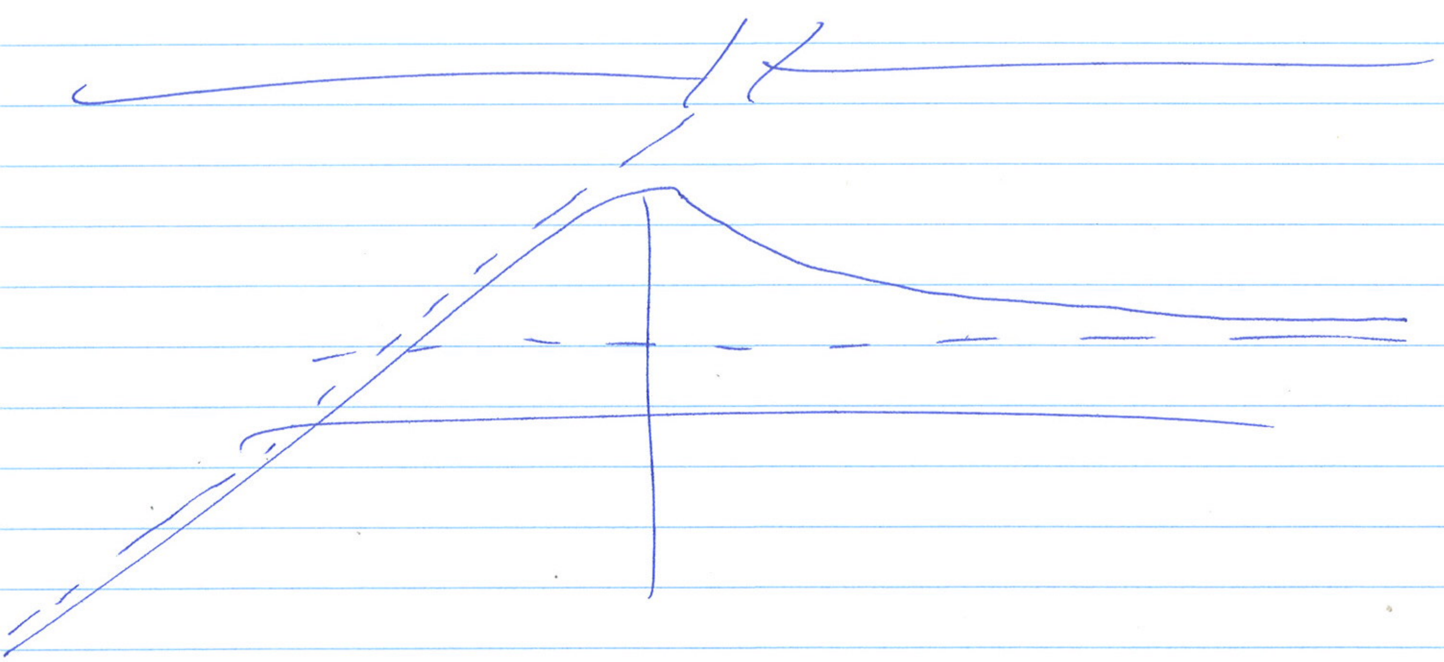
	$(-\infty, 0)$	$(0, \infty)$
$f(x)$	C.D.	C.U.
$f''(x)$	<0	>0

↑ inflection point at $x = 0$.





Try showing $y = \sqrt{x^2+4x}$ has slant asymptotes $\begin{cases} y = x+2 \\ y = -x-2. \end{cases}$

later.

§ 4.4 L'Hôpital's Rule

To find challenging limits (say for curve sketching):

$$\lim_{x \to 0} \frac{\sin x}{x} = ?.$$

Think back to Taylor Series:

$$\sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

around $a = 0$. $f'(0)x$.

$$\lim_{x \to 0} \frac{\sin x}{x} \simeq \lim_{x \to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}{x}$$

$$= \lim_{x \to 0} \frac{x}{x} - \frac{x^3}{3!\,x} + \frac{x^5}{5!\,x} + \cdots$$

$$= 1 + 0 + 0 + 0 \cdots$$

$$= 1.$$

Somehow the derivative is important:

L'Hôpital's Rule: For $\lim_{x \to a} \frac{f(x)}{g(x)}$.

if the limit is of the form

"$\frac{0}{0}$" or "$\frac{\infty}{\infty}$".

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Example: $\lim_{x \to 0} \frac{\sin x}{x} \stackrel{LH}{=} \lim_{x \to 0} \frac{\cos x}{1}$

$= 1$.

Clicker Q: Which can we not use L.H on:

A) $\lim_{x \to 1} \frac{\ln x}{1-x}$ B) $\lim_{x \to \infty} \frac{e^x}{x^2}$

C) $\lim_{x \to 0} \frac{\tan x - x}{x^3}$ D) $\lim_{x \to \pi^-} \frac{\sin x}{1-\cos x}$.

$\quad \quad$ $= 0$.

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B) $\lim_{x \to \infty} \frac{e^x}{x^2} \overset{LH}{=} \lim_{x \to \infty} \frac{e^x}{2x}$.

$\overset{LH}{=} \lim_{x \to \infty} \frac{e^x}{2} = \infty$.

Example: Clicker Q:

$$\lim_{x \to 0} \frac{\ln(1-x) - \sin x}{1 - \cos^2 x}$$

A) 0 B) 1

C) ∞ D) ~~N.D.E.~~ D.N.E.

E) None of the above.

$$\overset{LH}{=} \lim_{x \to 0} \frac{\frac{-1}{1-x} - \cos x}{+2\cos x \sin x}$$

$$= \lim_{x \to 0} \frac{\frac{-1}{1-x}}{2\cos x \sin x} + \lim_{x \to 0} \frac{-\cancel{\cos} x}{2\cancel{\cos} x \sin x}.$$

$$= \text{“} -1/0 \text{''} + \frac{-1}{2\sin x} = \text{“}\frac{-2}{0}\text{''}$$

$$= -\infty.$$