① March 26

Midterm Session will be posted as a pencast?

§ 4.4 L'Hôpital's Rule

Sometimes we can make limits look like the form "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$".

Example! $\lim_{x\to 0^+} x\ln x$. ("$0\cdot\infty$")

We can't apply L'Hôpital directly.

ClickerQ: A) $\dfrac{\ln x}{1/x}$ B) $\dfrac{x}{1/\ln x}$.

Is there a better Choice?

A) $\displaystyle\lim_{x\to 0^+}\frac{\ln x}{1/x} \overset{LH}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}\frac{1}{x}\cdot\frac{-x^2}{1}$

$= \lim_{x\to 0^+} -x$

$= 0$.

B) $\displaystyle\overset{LH}{=} \lim_{x\to 0^+}\frac{1}{-(\ln x)^{-2}/x}$.

$(\ln x)^{-1} \longrightarrow \dfrac{(\ln x)^{-2}}{x}$.

$= \lim_{x\to 0^+}$ ......

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Example: $\lim_{x\to 0^+} x^x$ $\left(\text{"} 0^0 \text{"}\right)$

Recall: $e^{\ln x} = x$

So $x^x = \left(e^{\ln x}\right)^x$.

$= e^{x\ln x}$.

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1$$

Since $\lim_{x\to 0^+} x\ln x = 0$.

Example: Sketch the graph of $f(x) = xe^x$.

1. Domain: $\mathbb{R}$
2. Intercepts: x-int: 0, y-int: 0.
3. Asymptotes; Horizontal.

$\lim_{x\to\infty} xe^x = \infty$ No asymptote.

$\lim_{x\to -\infty} xe^x = \text{"} -\infty \cdot 0 \text{"}$

Clicker Q: A) $\lim_{x\to -\infty} \frac{x}{e^{-x}}$ B) $\lim_{x\to -\infty} \frac{e^x}{1/x}$.

B) $\overset{LH}{=} \lim_{x \to -\infty} \frac{e^x}{-1/x^2} = \lim_{x \to -\infty} -x^2 e^x.$

$= \text{"} -\infty \cdot 0 \text{"}.$

A) $\overset{LH}{=} \lim_{x \to -\infty} \frac{1}{-e^{-x}} = \lim_{x \to -\infty} -e^x$

$= -0. = 0.$

$\Longrightarrow$ Horizontal Asymptote $y = 0$.

4. $f(x) = xe^x$. (Inc/Dec)

$f'(x) = e^x + xe^x$

$= e^x(1+x)$

Critical point: $x = -1$.

	$(-\infty, -1)$	$(-1, \infty)$
$f(x)$	$\searrow$	$\nearrow$
$f'(x)$.	<0	>0

local min at $x = -1$.

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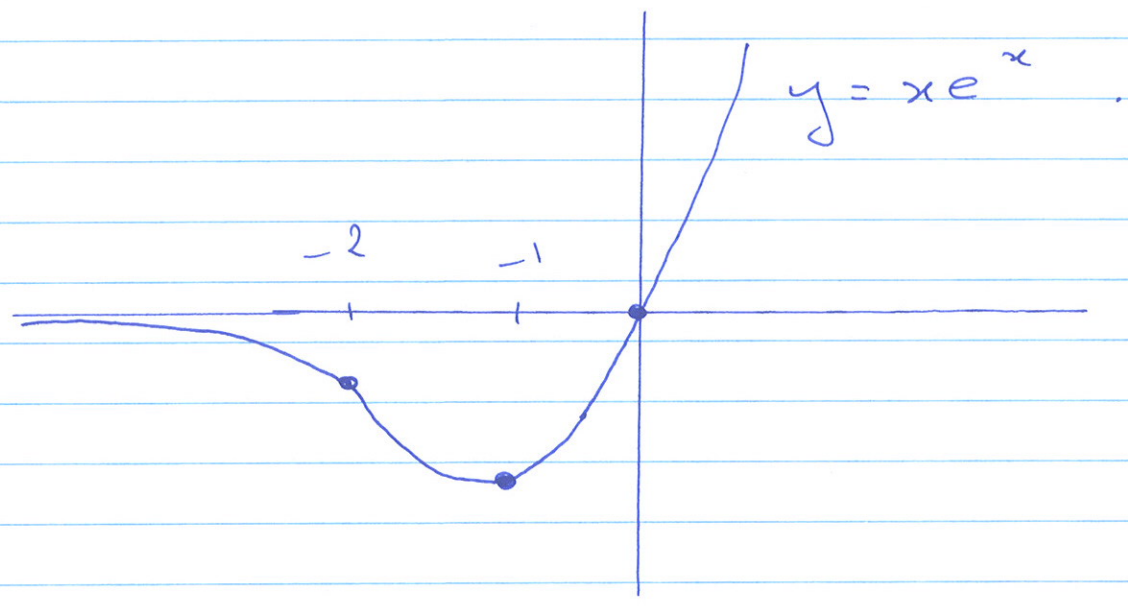
5. Concavity.

$$f''(x) = e^x + e^x + xe^x$$
$$= 2e^x + xe^x$$
$$= e^x(2+x).$$

Possible inflection point: $x = -2$.

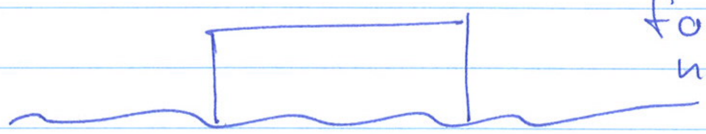
	$(-\infty, -2)$	$(-2, \infty)$
$f(x)$	C. D	C. U.
$f''(x)$	<0	>0

inflection point at $x = -2$.



§ 4.7 Optimization

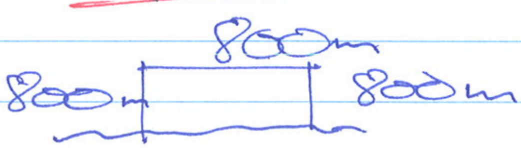
Example: A farmer has 2400m of fencing and wants to fence off a rectangular field that borders a straight river.



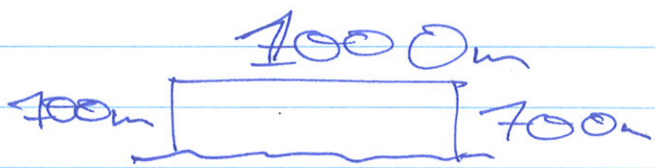
farmer needs no fence along river.

What are the dimensions of the field that has the largest area?

Possible Configurations:

- 

  $A = 640{,}000\,m^2$

- 

  $A = 700{,}000\,m^2$.

Solution: Let A be the area.
Let x and y the depth and width.

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1 ... $A = xy$

2 ... $2x + y = 2400$.

Solve for y in 2.

$$y = 2400 - 2x.$$

Substitute to 1.

$$A(x) = x(2400 - 2x).$$
$$= 2400x - 2x^2.$$

We have the zeros: $\begin{cases} A(0) = 0 \\ A(1200) = 0. \end{cases}$

So, consider $A(x)$ on $[0, 1200]$.

Now the problem is just to find absolute max.

$A'(x) = 2400 - 4x$ (exist everywhere)

$A'(x) = 0$ when $x = 600$ m

critical point.

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$$y = 2400 - 2(600) = 1200$$

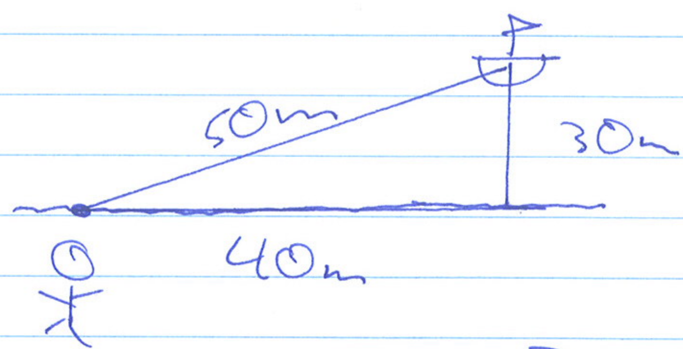
$$A(600) = 600 \cdot 1200 = 720{,}000 \text{m}^2.$$

The field should be 600m by 1200m.

Problem Solving Steps:

1. Pictures, Examples, Understand.
2. Notation.
3. Find relationship(s) between relevant variables.
4. Find ~~an~~ ~~equation~~ a function of one variable on some domain.
5. Use Calculus to find abs. max/min.
6. Reflect.

Example! Friend in a boat.



Reach the boat as fast as possible.

You can run at 3m/s.
Swim at 1m/s.

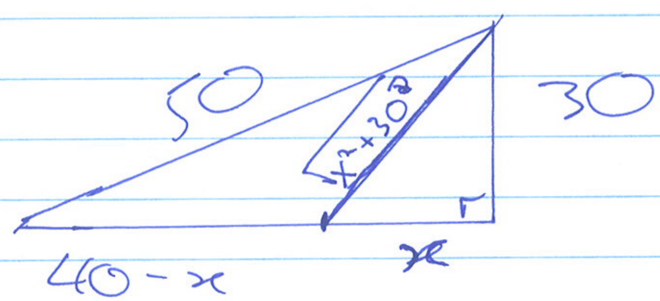
We try to minimize time.

Recall: speed $= \dfrac{\text{distance}}{\text{time}} \Rightarrow \text{time} = \dfrac{\text{distance}}{\text{speed}}$

Just swim: $T = \dfrac{50\text{m}}{1\text{m/s}} = 50\text{s}.$

Run 40m
Swim 30m

$$T = \text{time run} + \text{time swim}$$
$$= \frac{40\text{m}}{3\text{m/s}} + \frac{30\text{m}}{1\text{m/s}}.$$
$$= \left(13\tfrac{1}{3} + 30\right)\text{s}$$
$$= 43\tfrac{1}{3}\ \text{s}.$$



$$T = \text{Time Run} + \text{Time Swim}$$
$$= \frac{40-x}{3} + \frac{\sqrt{x^2+30^2}}{1}.$$

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$$T(x) = \frac{40-x}{3} + (x^2+30^2)^{1/2}$$

on $[0, 40]$.

Find abs. min.

Just swim $T(40) = 50s$

$T(0) = 43^{1/3}$ s.

Use closed interval method.

$$T'(x) = -\frac{1}{3} + \frac{1}{\cancel{2}}(x^2+30^2)^{-1/2} \cdot \cancel{2}x$$

$$= \frac{x}{\sqrt{x^2+30^2}} - 1/3 .$$

Exists everywhere.

$$T'(x) = 0 \Rightarrow \frac{1}{3} = \frac{x}{\sqrt{x^2+30^2}} .$$

$$\Rightarrow \sqrt{x^2+30^2} = 3x . \quad *$$

$$x^2+30^2 = 9x^2$$

$$30^2 = 8x^2$$

$$x^2 = \frac{30^2}{8}$$

$$x = \frac{30}{\sqrt{8}} \text{ m} \simeq 10.6 \text{m} .$$

(10).

$$T\left(\frac{30}{\sqrt{8}}\right)$$

$$= \frac{40 - \frac{30}{\sqrt{8}}}{3} + \underbrace{\sqrt{\left(\frac{30}{\sqrt{8}}\right)^2 + 30^2}}_{\text{from (*) is } 3\left(\frac{30}{\sqrt{8}}\right)}.$$

$$= \frac{40 - \frac{30}{\sqrt{8}}}{3} + 3\left(\frac{30}{\sqrt{8}}\right).$$

$$= \frac{40}{3} - \frac{10}{\sqrt{8}} + \frac{90}{\sqrt{8}}$$

$$= \frac{40}{3} + \frac{80}{\sqrt{8}} = \frac{40}{3} + \frac{80}{2\sqrt{2}}$$

$$= \frac{40}{3} + \frac{40}{\sqrt{2}}.$$

$$\simeq 41.65.$$

$$T\left(\frac{30}{\sqrt{8}}\right) = \frac{40}{3} + \frac{40}{\sqrt{2}} \overset{?}{<} \frac{40}{3} + 30 = T(0)$$

$$\frac{4\cancel{0}}{\sqrt{2}} \overset{?}{<} 3\cancel{0}. \quad \frac{4}{\sqrt{2}} \overset{?}{<} 3$$

$$\boxed{16 < 18} \Leftarrow \frac{16}{2} \overset{?}{<} 9.$$