① March 31

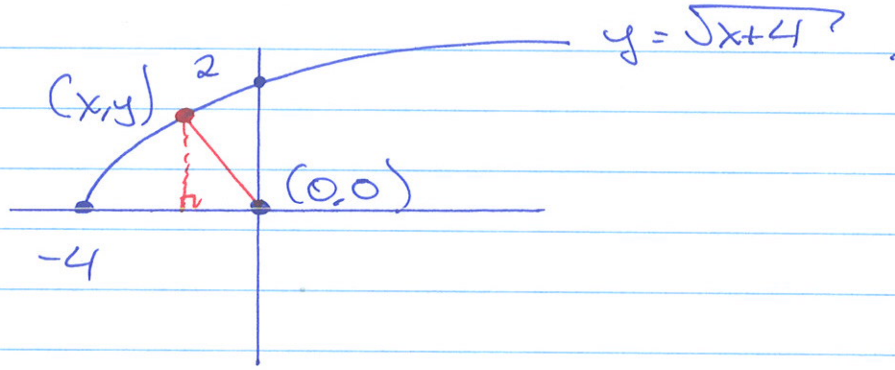
## §4.7 Optimization

Last class: Closed interval method.
Today we will see first derivative test.

**Example:** Find the point on the curve $y = \sqrt{x+4}$ that is closest to the point $(0,0)$.

Clickers:
A: Ok
B: Confused
C: Bored.

1. Picture.
2. Notation




3. Equation

$$d = \sqrt{(x-0)^2 + (y-0)^2}$$

$$d = \sqrt{x^2 + y^2}.$$

4. One-variable. Use $y = \sqrt{x+4}$

$$d(x) = \sqrt{x^2 + (\sqrt{x+4})^2}$$

$$= \sqrt{x^2 + x + 4}.$$

Domain: $x \in [-4, \infty)$

5. Calculus. Use First Derivative Test.

$$d'(x) = \frac{1}{2}\left(x^2 + x + 4\right)^{-1/2}(2x+1)$$

$$= \frac{2x+1}{2\sqrt{x^2+x+4}}$$ exists on $[-4, \infty)$.

Critical points: $d'(x) = 0$ when

$$2x + 1 = 0$$
$$x = -1/2.$$

| | $(-4, -1/2)$ | $(-1/2, \infty)$ |
|---|---|---|
| $d(x)$ | $\searrow$ | $\nearrow$ |
| $d'(x)$ | $<0$ | $>0$ |

local min.

So local min at $x = -1/2$.
Since there is only one critical point and our function goes from dec. to inc. the point

$(-1/2, \sqrt{3.5})$ is a global min.

(3) Clickers: A: Ok C: bored.
B: lost

Example: The energy required for a fish to swim a distance $L$ at speed $v$ against a current of speed $u$ is:

$$E(v) = av^3 \frac{L}{v-u} \qquad (u<v)$$

where $a$ is a positive constant. How fast should the fish swim to minimize energy?

Domain: $v \in (u, \infty)$

Find critical points: $E(v) = \dfrac{aLv^3}{v-u}$.

$$E'(v) = \frac{3aLv^2(v-u) - 1(aLv^3)}{(v-u)^2}.$$

$$= \frac{3aLv^3 - 3aLv^2u - aLv^3}{(v-u)^2}.$$

$$= (2aLv^3 - 3aLv^2u)/(v-u)^2.$$

$$= \frac{aLv^2(2v-3u)}{(v-u)^2}.$$

no problem in denominator.

So, $E'(v) = 0$ when

$$0 = \underbrace{aLv^2}_{\neq 0}\underbrace{(2v - 3u)}_{= 0}.$$

$$2v - 3u = 0$$
$$2v = 3u$$
$$v = \frac{3}{2}u$$

critical point.

Is it a min? local min.

| | $(u, \frac{3}{2}u)$ | $(\frac{3}{2}u, \infty)$ |
|---|---|---|
| $E(v)$ | $\searrow$ | $\nearrow$ |
| $E'(v)$ | $<0$ | $>0$ |

For $(u, \frac{3}{2}u)$ is

Clickers:

A: $E'(v) > 0$

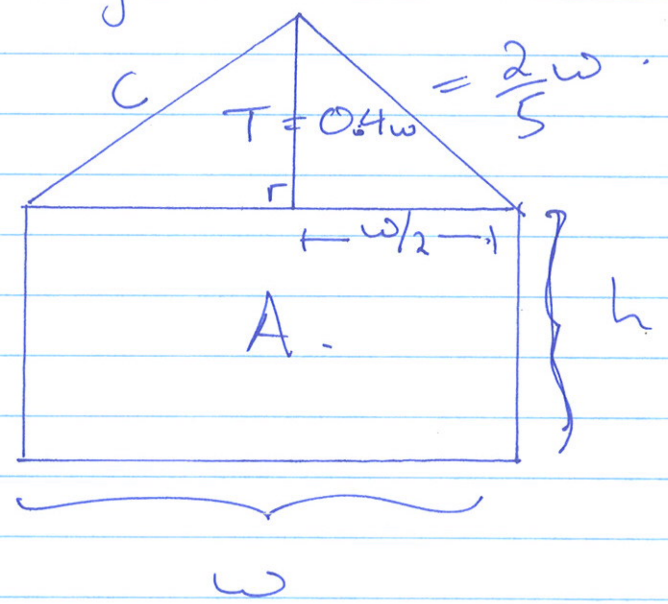
B: $E'(v) < 0$.

Only one critical point. $v = \frac{3}{2}u$ is absolute min.

The fish should swim at 3/2 times the speed of the current.

Clickers: A: ok C: bored.
B: lost

Example: Consider a window the shape of which is a rectangle of height $h$ surmounted by a triangle having height $T$, that is $0.4$ times the width, $w$, of the rectangle. Minimize perimeter given a fixed Area.




1. Picture
2. Notation.

Minimize Perimeter.
A - constant.

3. Equation(s)

$$P = 2c + 2h + w.$$

4. One variable: Want $P(w)$.

$$c^2 = \left(\frac{w}{2}\right)^2 + \left(\frac{2}{5}w\right)^2.$$

$$c^2 = \frac{w^2}{4} + \frac{4w^2}{25}$$

$$= \frac{25w^2}{100} + \frac{16}{100}w^2.$$

$$c = \frac{\sqrt{41}\,w}{10}. \qquad = 41w^2/100$$

(B)

$$A = wh + \cancel{2}\ \frac{1}{2} w \cdot \frac{2}{5} w$$

$$A = wh + \frac{w^2}{5}.$$

$$wh = A - \frac{w^2}{5}$$

$$h = \frac{A}{w} - \frac{w}{5}.$$

$$P = 2c + 2h + w$$

$$P(w) = 2\,\frac{\sqrt{41}}{10} w + 2\left(\frac{A}{w} - \frac{w}{5}\right) + w.$$

$$= \frac{\sqrt{41}}{5} w + \frac{2A}{w} - \frac{2}{5} w + w$$

$$= \frac{\sqrt{41}}{5} w + \frac{3}{5} w + \frac{2A}{w}.$$

$$P(w) = \frac{\sqrt{41}+3}{5} w + \frac{2A}{w}$$

Domain! $w \in (0, \infty)$.

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5. Calculus. $P'(\omega) = \dfrac{\sqrt{41}+3}{5} - 2A\omega^{-2}$.

$P'(\omega) = 0$ : $\dfrac{2A}{\omega^2} = \dfrac{\sqrt{41}+3}{5}$.

$$\omega^2 = \frac{2A \cdot 5}{\sqrt{41}+3}$$

critical point.

$$\omega = \sqrt{\frac{10A}{\sqrt{41}+3}}.$$

| | $\left(0, \sqrt{\frac{10A}{\sqrt{41}+3}}\right)$ | $\left(\sqrt{\frac{10A}{\sqrt{41}+3}}, \infty\right)$ |
|---|---|---|
| $P(\omega)$ | $\searrow$ | $\nearrow$ |
| $P'(\omega)$ | $<0$ | $>0$ |

$$P'(\omega) = \frac{\sqrt{41}+3}{5} - \frac{2A}{\omega^2}.$$

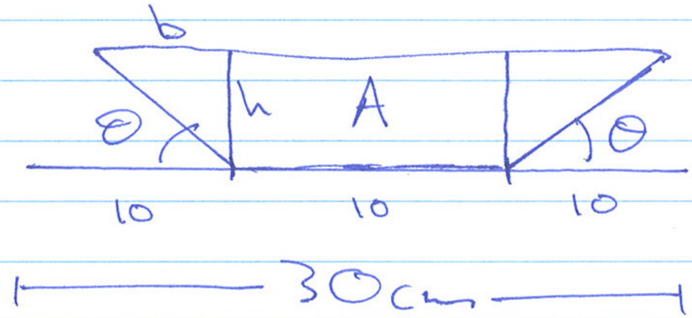
Only one critical point

$\Rightarrow$ global min.

<u>Example:</u> A rain gutter is to be constructed from a metal sheet of width 30cm by bending up one-third of the sheet on each side through an angle $\theta$.

How should $\theta$ be chosen so that the gutter will carry the maximum amount of water?

1. Picture
2. Notation




Choose $\theta$ to maximize area.

3. Equation:

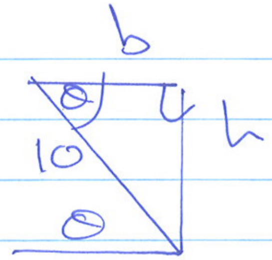
$$\text{Area} = 10h + 2\left(\frac{bh}{2}\right).$$

$$= 10h + bh$$

Get rid of $h$ and $b$ in favour of $\theta$.

## 4. One Eq.

SOHCAHTOA.




$$\sin\theta = \frac{h}{10} \qquad \cos\theta = \frac{b}{10}$$

$$\Rightarrow \quad h = 10\sin\theta$$
$$b = 10\cos\theta.$$

$$A(\theta) = 10\cdot 10\sin\theta + 10\cdot 10\cos\theta\sin\theta$$

$$A(\theta) = 100\sin\theta + 100\cos\theta\sin\theta.$$

Domain: $\theta \in [0, \pi/2]$..

## 5. Closed interval method.

$$A'(\theta) = 100(\cos\theta - \sin^2\theta + \cos^2\theta)$$

$A'(\theta) = 0$ when:

$$0 = \cos\theta - \sin^2\theta + \cos^2\theta$$

Recall: $\sin^2\theta + \cos^2\theta = 1$ so,

$$\begin{aligned} 0 &= \cos\theta - 1 + \cos^2\theta + \cos^2\theta \\ &= 2\cos^2\theta + \cos\theta - 1 \\ &= \underbrace{(2\cos\theta - 1)}_{0''}\underbrace{(\cos\theta + 1)}_{\text{gives } \theta = \pi \text{ : outside domain.}} \end{aligned}$$

$$2\cos\theta - 1 = 0$$

$$\cos\theta = 1/2. \Rightarrow \theta = \pi/3. \leftarrow \text{critical point.}$$

Endpoint: $\theta = 0, \pi/2$.

$A(0) = 0$, $\quad A(\pi/3) = 100\sin(\pi/3) + 100\cos(\pi/3)\sin(\pi/3)$

$A(\pi/2) = 100$, $\quad = 100\cdot\frac{\sqrt{3}}{2} + 100\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}.$

(10)

$$A(\pi/3) = 50\sqrt{3} + 25\sqrt{3}$$
$$= 75\sqrt{3}$$

$$75\sqrt{3} \quad \text{vs.} \quad 100$$

$$\cancel{(75\sqrt{3})^2 \quad \text{vs} \quad 100^2}$$

OR/

$$\left(\frac{3}{4}\sqrt{3}\right)^2 \quad \text{vs.} \quad (1)^2$$

$$\frac{9}{16} \cdot 3 \quad \text{vs.} \quad 1.$$

$$\frac{27}{16} > 1$$

So $\quad 75\sqrt{3} > 100.$

Therefore, the max area is $75\sqrt{3}\ \text{cm}^2$ and is achieved at an angle of $\theta = \pi/3$ radians.