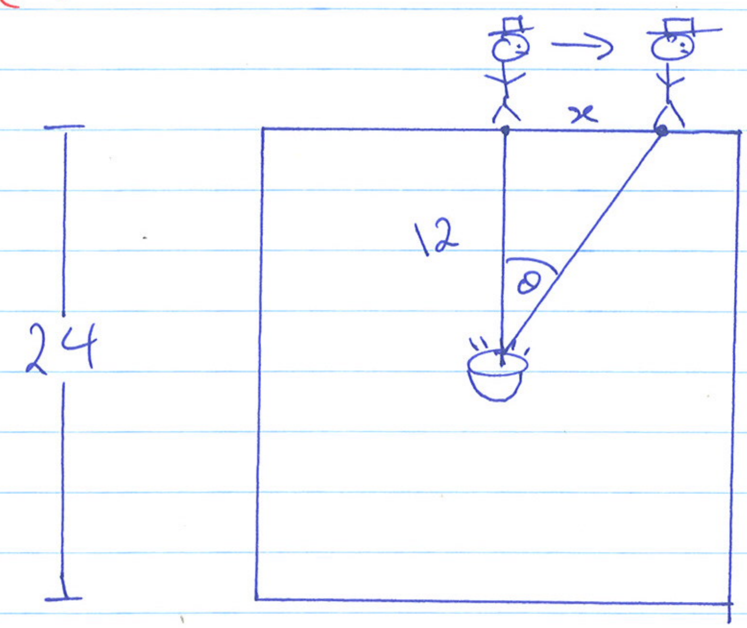
Example: A man walks along the edge of a square field with side length 24m. A spotlight is positioned in the middle of the field. When he reaches the point on his path that is closest to the light it turns on. He begins to run at 2m/s. While doing so the spotlight follows him. How fast is the spotlight rotating after 2.5 seconds?

1. Picture.



2. Notation. Given/Required rate.

x: distance from the centre of the edge to the man.
θ: angle the beam makes with the vertical.

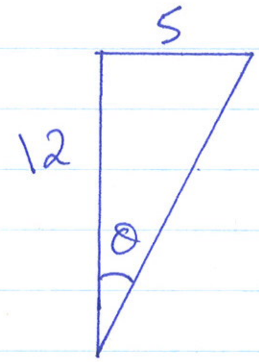
②

In terms of this notation we have
$\frac{dx}{dt} = 2$ and need $\frac{d\theta}{dt}$.

We can calculate x when $t = 2.5$
using
distance = speed · time

$$x = 2 \cdot 2.5$$
$$= 5.$$

So, at $t = 2.5$ our picture looks like this

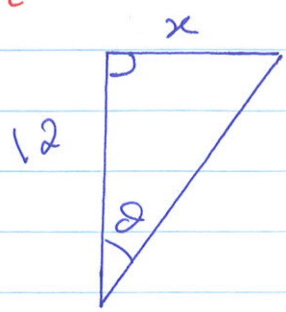


This picture gives a snapshot of our system at a particular time. However, no information can be gleaned about $d\theta/dt$.

Our goal is to relate $\frac{d\theta}{dt}$ and $\frac{dx}{dt}$.

We need an equation involving θ and x while thinking about θ and x as functions of t.

3. Equation



$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{x}{12}.$$

4. Chain Rule

Take $\frac{d}{dt}$ of both sides.

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{x}{12}\right)$$

So,
$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{12}\frac{dx}{dt}.$$

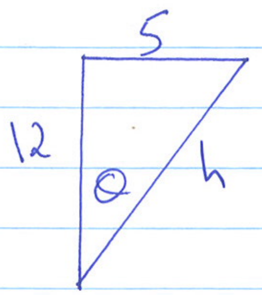
5. Solve/Substitute

2m/s

$$\frac{d\theta}{dt} = \frac{1}{12}\cos^2\theta\,\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{2}{12}\cos^2\theta$$

We still require the angle θ when $t=2.5$.
Actually, we need only find $\cos\theta$.



$$\cos\theta = \frac{12}{h}, \quad h = \sqrt{12^2+5^2} = \sqrt{144+25} = \sqrt{169} = 13.$$

④

So, $\cos\theta = \frac{12}{13}$ when $t = 2.5$.

Now, $\frac{d\theta}{dt} = \frac{2}{12}\cos^2\theta$

$= \frac{2}{12}\left(\frac{12}{13}\right)^2 = \frac{2 \cdot 12}{169} = \frac{24}{169}$ radians/s.

This is about 8.14 degrees/s.

6. Reflect. Before Substitution: $\frac{d\theta}{dt} = \frac{1}{6}\cos^2\theta$.

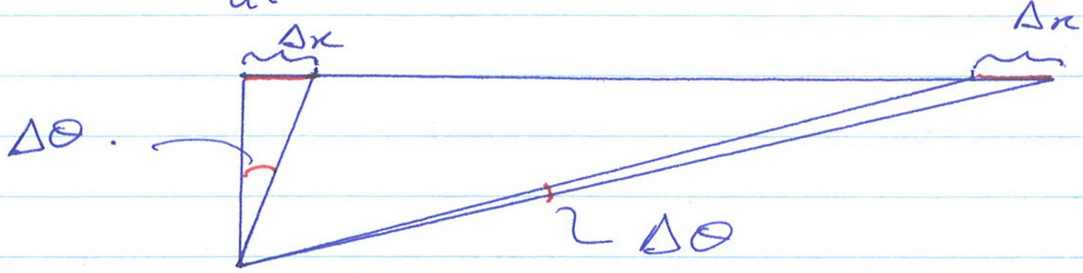
When θ is small, $\cos^2\theta \simeq 1$

So, $\frac{d\theta}{dt} \simeq \frac{2}{12} = 1/6$.

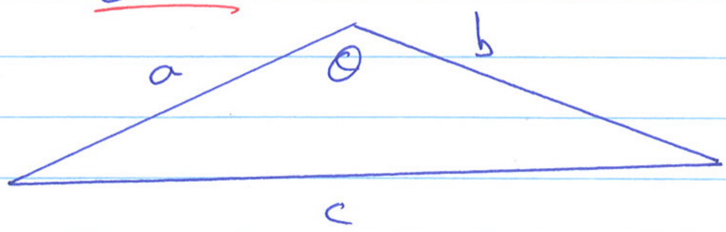
When θ is "large" or close to $\pi/2$, $\cos^2\theta \simeq 0$

So,

$\frac{d\theta}{dt}$ is small.



For the next problem we will need Cosine Law

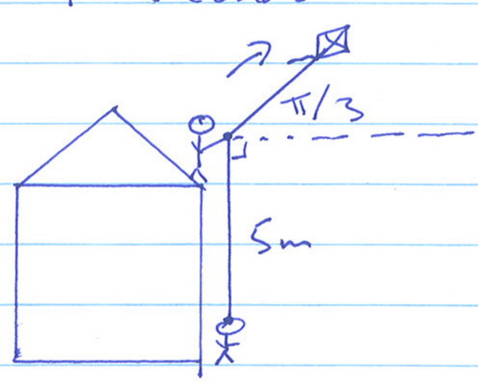


$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Cosine Law is like the Pythagorean Theorem for non-right angled triangles.

Note, if $\theta = \pi/2$ then $c^2 = a^2 + b^2$.

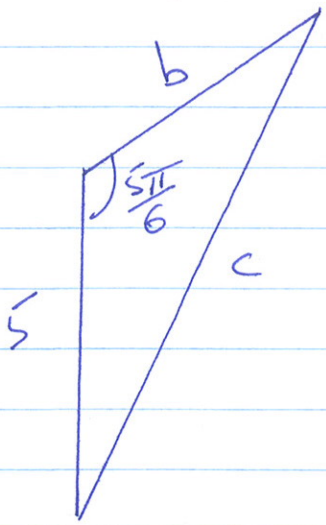
Example: Your friend stands on a roof flying a kite. You stand on the ground below observing. Initially your friend holds the kite 5m above your eyes. The kite rises with speed 0.5m/s in such a way that its string makes an angle of $\pi/3$ with the horizontal. How fast is the distance between you and the kite changing after 4 seconds?



⑥

So, consider the following triangle.

1. Picture.



2. Notation. Given/Required

b is the distance from the kite to your friend.
c is the distance from the kite to you.

We have $\frac{db}{dt}$ and need $\frac{dc}{dt}$.

We think of b and c to be functions of time and consider the following Equation (3.)

$$c^2 = 5^2 + b^2 - 2\cdot 5\, b \cos\left(\frac{5\pi}{6}\right)$$

$$c^2 = 25 + b^2 + \cancel{2}\cdot 5\, b \frac{\sqrt{3}}{\cancel{2}}$$

Now take d/dt using Chain Rule! (4)

$$2c\frac{dc}{dt} = 2b\frac{db}{dt} + 5\sqrt{3}\frac{db}{dt}.$$

It remains to Solve and Substitute. (5).