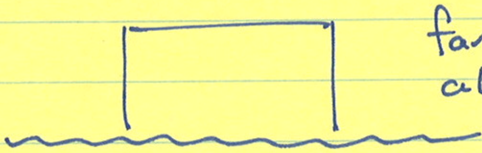
Optimization

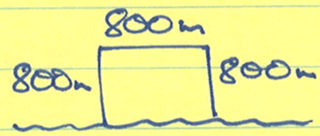
Example: A farmer has 2400m of fencing and wants to fence off a rectangular field that borders a straight river.



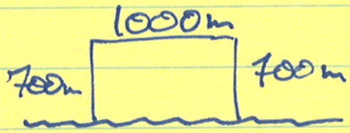
farmer needs no fence along the river.

What are the dimensions of the field thas has the largest area?

Possible Configurations!

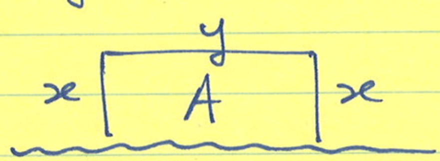


$A = 800m \cdot 800m$
$= 640{,}000 m^2$



$A = 1000m \cdot 700m$
$= 700{,}000 m^2$.

Solution: Let A be the area of the field.
Let x and y be the depth and width.



We should expect y to be larger than x to take advantage of the fact that we get the side opposite y for free.

We know, ① ... $A = xy$; area of a rectangle
② ... $2400 = 2x + y$; length of fencing.

We can solve for y in the second to get, $y = 2400 - 2x$.

③

Substituting to ① gives,

$$A(x) = x(2400 - 2x) = 2400x - 2x^2 .$$

We have the zeros, $A(0) = 0$, $A(1200) = 0$.
So, consider $A(x)$ on the interval $[0, 1200]$.

We seek critical points. The derivative is given by,
$A'(x) = 2400 - 4x$ and exists everywhere.

$$A'(x) = 0 \iff 2400 - 4x = 0$$
$$4x = 2400$$
$$x = 600\,m, \text{ our critical point.}$$

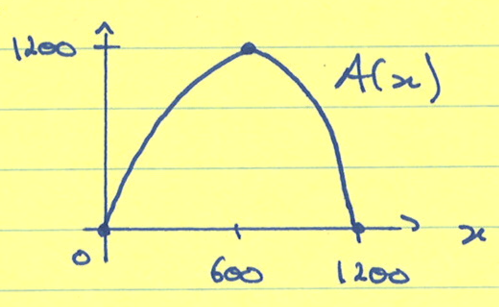
The corresponding y is given by, $y = 2400 - 2(600)$
$= 1200\,m$,

and,

$$A(600m) = 1200\,m \cdot 600\,m = 720{,}000\,m^2 .$$

The field should be 600m deep and 1200m wide.

Note that $A''(x) = -4 < 0$ for all x so $A(x)$ is always concave down.



Problem Solving for Optimization

1. Understand the problem. Pictures, Examples.

2. Introduce notation. Identify quantity to be optimized.

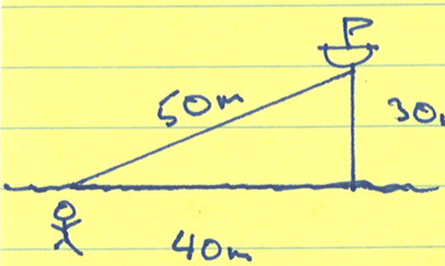
3. Find relationship(s) between all relevant variables.

4. Use information from the problem to find an equation relating the desired quantity and only one other variable, on some domain.

5. Use calculus to find absolute max/min.

6. Reflect. Is the answer reasonable? Units?

Example 2: Friend in a boat.



Reach your friend as fast as possible. You can run at 3m/s and swim at 1m/s.

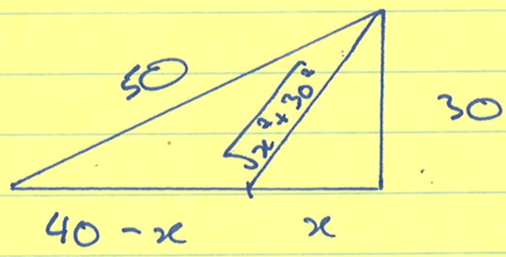
Recall: $\text{speed} = \dfrac{\text{distance}}{\text{time}}$ so $\text{time} = \dfrac{\text{distance}}{\text{speed}}$.

Just swim: $T = \dfrac{50m}{1^{m}/s} = 50s$.

Run 40m,
Swim 30m: $T = \dfrac{40m}{3^{m}/s} + \dfrac{30m}{1^{m}/s} = 13\tfrac{1}{3}s + 30s = 43\tfrac{1}{3}s$.

5

Let x be the distance indicated.



T = Time Run + Time Swim

$$= \frac{(40-x)}{3} + \frac{\sqrt{x^2+30^2}}{1}.$$

Take, $T(x) = \dfrac{40-x}{3} + (x^2+30^2)^{1/2}$ on $[0,40]$.

Now, $x=40$ corresponds to just swimming and $x=0$ to running $40m$ then swimming $30m$.

So, $\begin{cases} T(40m) = 50s \\ T(0m) = 43^{1/3}\,s \end{cases}$.

Since $T(x)$ is a continuous function on a closed interval the absolute minimum value exists and must be either a critical point or end point.

$$T'(x) = \frac{-1}{3} + \frac{1}{2}(x^2+30^2)^{-1/2}\,2x$$

$$= \frac{x}{\sqrt{x^2+30^2}} - \frac{1}{3},$$ which exists on $(0,40)$.
(in fact all x)

Example 2: (continued)

Critical points if $T'(x) = 0$.

$$T'(x) = 0 \iff 0 = \frac{x}{\sqrt{x^2+30^2}} - \frac{1}{3}$$

$$\Rightarrow \sqrt{x^2+30^2} = 3x \quad (*)$$
$$x^2+30^2 = 9x^2$$
$$8x^2 = 30^2$$
$$x^2 = \frac{30^2}{8}$$

So, $x = \frac{30}{\sqrt{8}} = \frac{30}{2\sqrt{2}} = \frac{15}{\sqrt{2}}\,m$.

Note, $x \geq 0$. $\cong 10.6\,m$

We want to know the value of $T(x)$ at our critical point.

$$T\left(\tfrac{15}{\sqrt{2}}m\right) = \frac{40 - 15/\sqrt{2}}{3} + \sqrt{\left(\tfrac{15}{\sqrt{2}}\right)^2 + 30^2}$$

$$= \frac{40}{3} - \frac{5}{\sqrt{2}} + 3\left(\frac{15}{\sqrt{2}}\right),$$ $15/\sqrt{2}$ was chosen to satisfy (*).

$$= \left(13\tfrac{1}{3} + \frac{40}{\sqrt{2}}\right)s \;.\; \cong 41.6s$$

Now,

$T\left(\frac{15}{\sqrt{2}}m\right) < T(0m) < T(40m)$, so the absolute minimum is at $x = 15/\sqrt{2}\,m$ with value $\left(13\tfrac{1}{3} + \frac{40}{\sqrt{2}}\right)s$.

Note, $T(15/\sqrt{2}) < T(0)$, since $40/\sqrt{2} < 30$,

since $\frac{10}{\sqrt{2}}(4) < \frac{10}{\sqrt{2}}(3\sqrt{2})$, since

$4 < 3\sqrt{2}$ since, $(4)^2 = 16 < 18 = (3\sqrt{2})^2$.