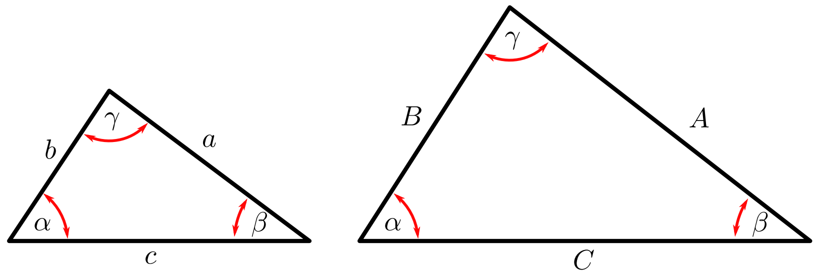
What you must remember

Similar triangles



Two triangles T_1, T_2 are similar when

- (AAA — angle angle angle) The angles of T_1 are the same as the angles of T_2.
- (SSS — side side side) The ratios of the side lengths are the same. That is

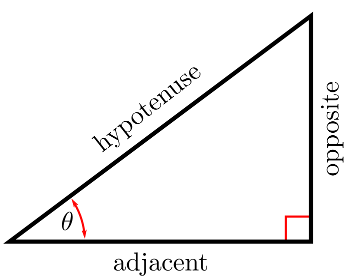
$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c}$$

- (SAS — side angle side) Two sides have lengths in the same ratio and the angle between them is the same. For example

$$\frac{A}{a} = \frac{C}{c} \text{ and angle } \beta \text{ is same}$$

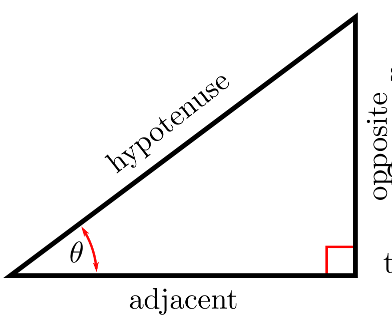
Pythagoras

For a right-angled triangle the length of the hypotenuse is related to the lengths of the other two sides by



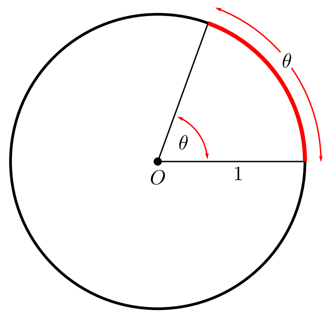
$$(\text{adjacent})^2 + (\text{opposite})^2 = (\text{hypotenuse})^2$$

Trigonometry — definitions



$$\begin{aligned} \sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} & \csc\theta &= \frac{1}{\sin\theta} \\ \cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}} & \sec\theta &= \frac{1}{\cos\theta} \\ \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} & \cot\theta &= \frac{1}{\tan\theta} \end{aligned}$$

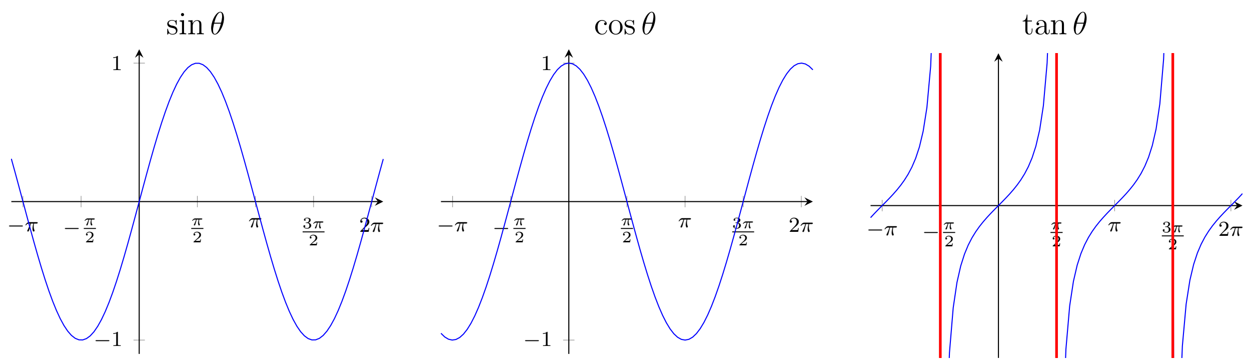
Radians, arcs and sectors



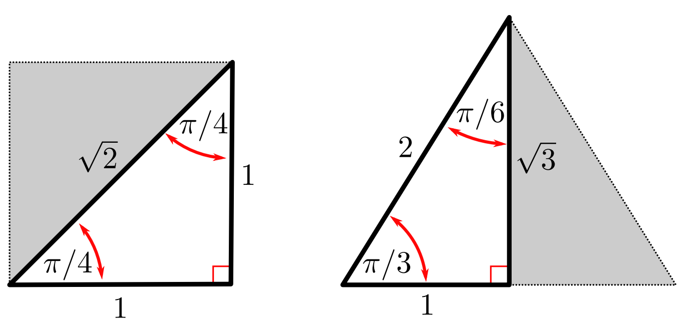
For a circle of radius r and angle of θ radians:

- Arc length $L(\theta) = r\theta$.
- Area of sector $A(\theta) = \frac{\theta}{2}r^2$.

Trigonometry — graphs



Trigonometry — special triangles



From the above pair of special triangles we have

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

Trigonometry — simple identities

- Periodicity

$$\sin(\theta + 2\pi) = \sin(\theta) \qquad\qquad \cos(\theta + 2\pi) = \cos(\theta)$$

- Reflection

$$\sin(-\theta) = -\sin(\theta) \qquad\qquad \cos(-\theta) = \cos(\theta)$$

- Rotation by $\pi/2$

$$\sin\left(\tfrac{\pi}{2} - \theta\right) = \cos\theta \qquad\qquad \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta$$

- Pythagoras

$$\sin^2\theta + \cos^2\theta = 1$$

Trigonometry — add and subtract angles

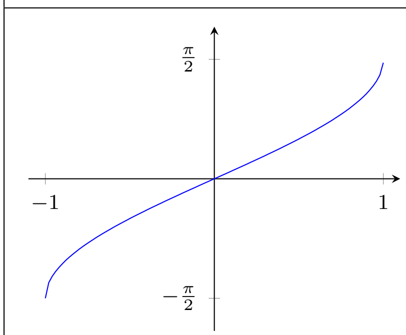
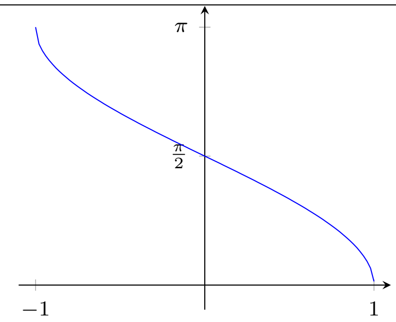
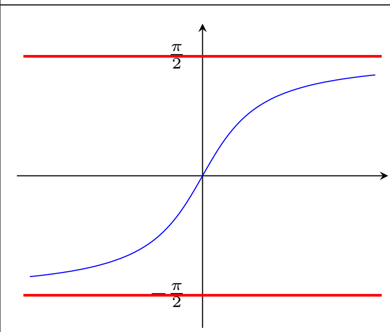
- Sine

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

- Cosine

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

Inverse trigonometric functions

$\arcsin x$	$\arccos x$	$\arctan x$
Domain: $-1 \le x \le 1$ Range: $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$	Domain: $-1 \le x \le 1$ Range: $0 \le \arccos x \le \pi$	Domain: all real numbers Range: $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$
		

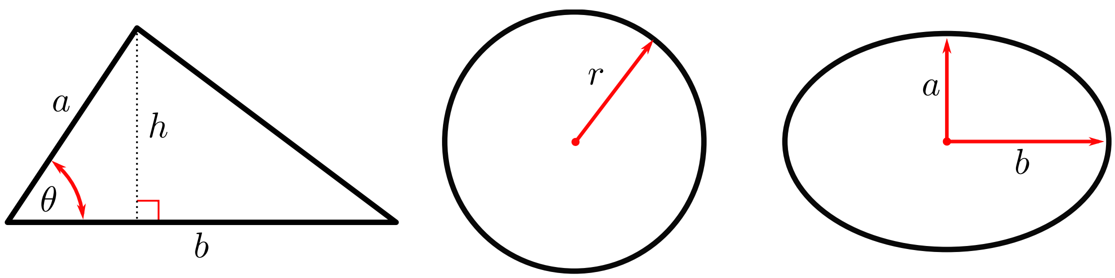
Since these functions are inverses of each other we have

$$\arcsin(\sin\theta) = \theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\arccos(\cos\theta) = \theta \qquad 0 \le \theta \le \pi$$
$$\arctan(\tan\theta) = \theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

and also

$$\sin(\arcsin x) = x \qquad -1 \le x \le 1$$
$$\cos(\arccos x) = x \qquad -1 \le x \le 1$$
$$\tan(\arctan x) = x \qquad \text{any real } x$$

Areas



- Area of a triangle

$$A = \frac{1}{2}bh = \frac{1}{2}ab\sin\theta$$

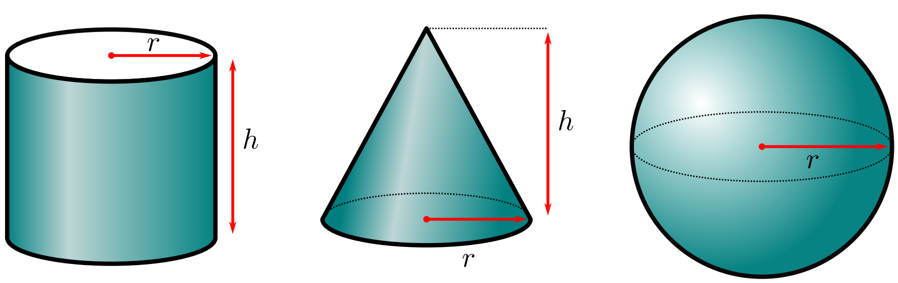
- Area of a circle

$$A = \pi r^2$$

- Area of an elipse

$$A = \pi ab$$

Volumes



- Volume of a cylinder

$$V = \pi r^2 h$$

- Volume of a cone

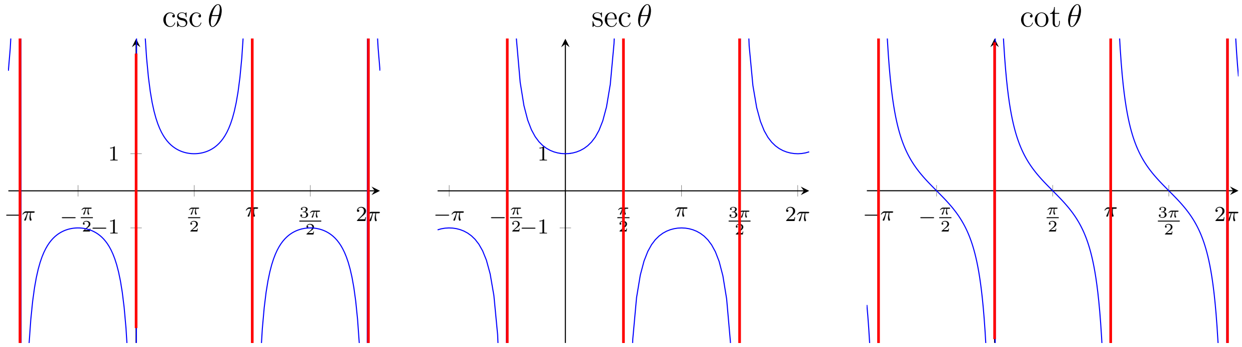
$$V = \frac{1}{3}\pi r^2 h$$

- Volume of a sphere

$$V = \frac{4}{3}\pi r^3$$

What you should be able to derive

- Graphs of $\csc\theta, \sec\theta$ and $\cot\theta$:



- More Pythagoras

$$\sin^2\theta + \cos^2\theta = 1 \xrightarrow{\text{divide by } \cos^2\theta} \tan^2\theta + 1 = \sec^2\theta$$

$$\sin^2\theta + \cos^2\theta = 1 \xrightarrow{\text{divide by } \sin^2\theta} 1 + \cot^2\theta = \csc^2\theta$$

- Sine — double angle (set $\beta = \alpha$ in sine angle addition formula)

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

- Cosine — double angle (set $\beta = \alpha$ in cosine angle addition formula)

$$\begin{aligned} \cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha) \\ &= 2\cos^2(\alpha) - 1 && (\text{use } \sin^2(\alpha) = 1 - \cos^2(\alpha)) \\ &= 1 - 2\sin^2(\alpha) && (\text{use } \cos^2(\alpha) = 1 - \sin^2(\alpha)) \end{aligned}$$

- Composition of trigonometric and inverse trigonometric functions:

$$\cos(\arcsin x) = \sqrt{1 - x^2} \qquad \sec(\arctan x) = \sqrt{1 + x^2}$$

and similar expressions.

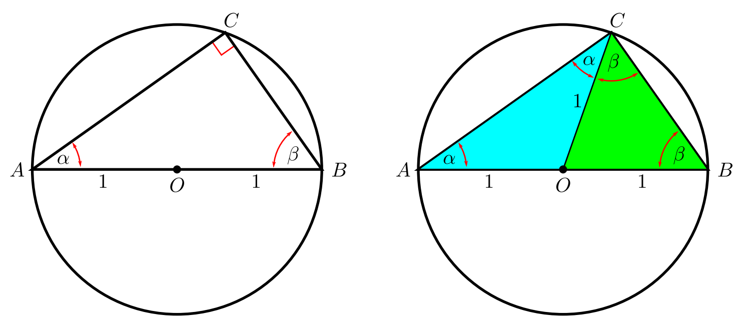
Stuff you don't need to know...

but perhaps you might like to know.

Where do these identities come from?

Thales' theorem

We want to get at right-angled triangles. A classic construction for this is to draw a triangle inside a circle, so that all three corners lie on the circle and the longest side forms the diameter of the circle. See the figure below in which we have scaled the circle to have radius 1 and the triangle has longest side 2.



Thales theorem states that the angle at C is always a right-angle. The proof is quite straightforward and relies on two facts:

- the angles of a triangle add to π, and
- the angles at the base of an isosceles triangle are equal.

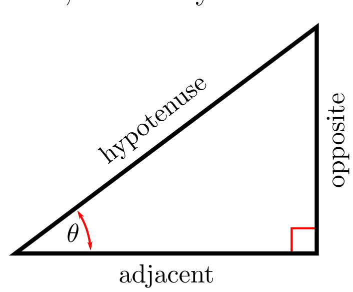
So we split the triangle ABC by drawing a line from the centre of the circle to C. This creates two isosceles triangles OAC and OBC. Since they are isosceles, the angles at their bases α and β must be equal (as shown). Adding the angles of the original triangle now gives

$$\pi = \alpha + (\alpha + \beta) + \beta = 2(\alpha + \beta)$$

So the angle at $C = \pi - (\alpha + \beta) = \pi/2$.

Pythagoras

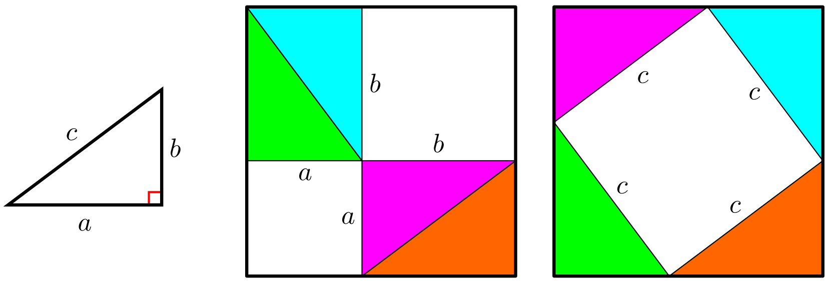
Since trigonometry, at its core, is the study of lengths and angles in right-angled triangles, we must include a result you all know well, but likely do not know how to prove.



The lengths of the sides of any right-angled triangle are related by the famous result due to Pythagoras

$$c^2 = a^2 + b^2.$$

There are many ways to prove this, but we can do so quite simply by studying the following diagram:



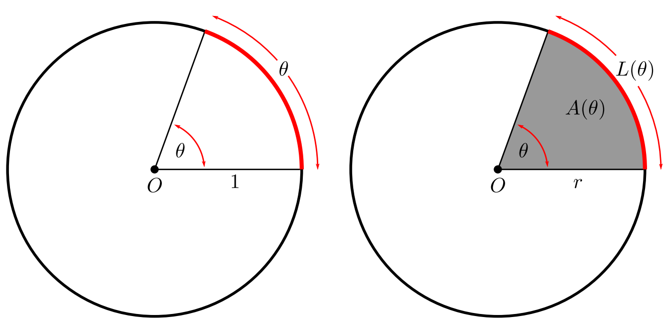
white area $= a^2 + b^2$ white area $= c^2$

We start with a right-angled triangle with sides labeled a, b and c. Then we construct a square of side-length $a + b$ and draw inside it 4 copies of the triangle arranged as shown in the centre of the above figure. The area in white is then $a^2 + b^2$. Now move the triangles around to create the arrangement shown on the right of the above figure. The area in white is bounded by a square of side-length c and so its area is c^2. The area of the outer square didn't change when the triangles were moved, nor did the area of the triangles, so the white area cannot have changed either. This proves $a^2 + b^2 = c^2$.

Trigonometry

Angles — radians vs degrees

For mathematics, and especially in calculus, it is much better to measure angles in units called radians rather than degrees. By definition, an arc of length θ on a circle of radius one subtends an angle of θ radians at the center of the circle.



The circle on the left has radius 1, and the the arc swept out by an angle of θ radians has length θ. Because a circle of radius one has circumference 2π we have

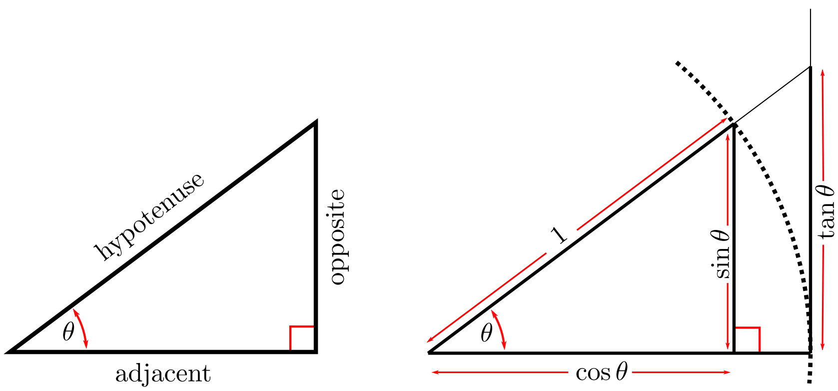
$$2\pi \text{ radians } = 360^\circ \qquad \pi \text{ radians } = 180^\circ \qquad {}^{\pi}/_{2} \text{ radians } = 90^\circ$$
$$\frac{\pi}{3} \text{ radians } = 60^\circ \qquad \frac{\pi}{4} \text{ radians } = 45^\circ \qquad \frac{\pi}{6} \text{ radians } = 30^\circ$$

More generally, consider a circle of radius r. Let $L(\theta)$ denote the length of the arc swept out by an angle of θ radians and let $A(\theta)$ denote the area of the sector (or wedge) swept out by the same angle. Since the angle sweeps out $\theta/2\pi$ of a whole circle we have

$$L(\theta) = 2\pi r \cdot \frac{\theta}{2\pi} = \theta r \qquad \text{and}$$
$$A(\theta) = \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{\theta}{2} r^2$$

Definitions

The trigonometric functions are defined as ratios of the lengths of the sides of a right-angle triangle as shown in the left of the diagram below . These ratios depend only on the angle θ.



sine

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

cosine

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

tangent

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta}$$

these are frequently abbreviated as

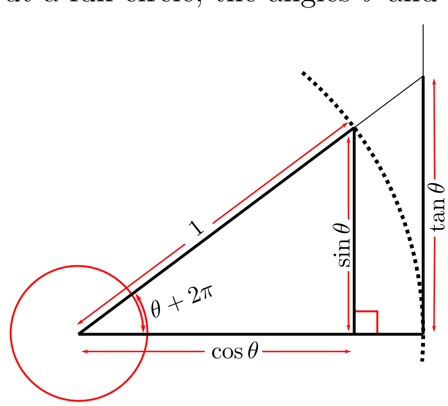
$$\sin\theta = \frac{\text{o}}{\text{h}} \qquad \cos\theta = \frac{\text{a}}{\text{h}} \qquad \tan\theta = \frac{\text{o}}{\text{a}}$$

which gives rise to the mnemonic

SOH CAH TOA

If we scale the triangle so that they hypotenuse has length 1 then we obtain the diagram on the right. In that case, $\sin\theta$ is the height of the triangle, $\cos\theta$ the length of its base and $\tan\theta$ is the length of the line tangent to the circle of radius 1 as shown.

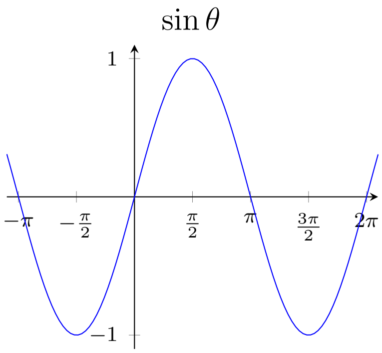
Since the angle 2π sweeps out a full circle, the angles θ and $\theta + 2\pi$ are really the same.

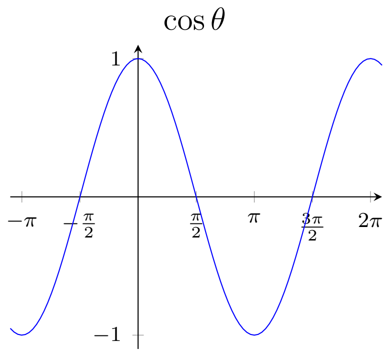


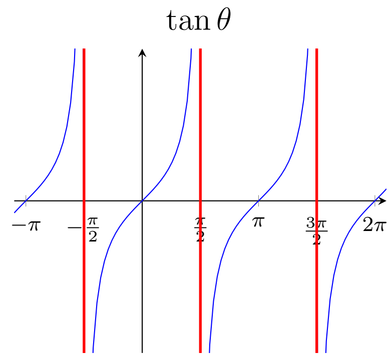
Hence all the trigonometric functions are periodic with period 2π. That is

$$\sin(\theta + 2\pi) = \sin(\theta) \qquad \cos(\theta + 2\pi) = \cos(\theta) \qquad \tan(\theta + 2\pi) = \tan(\theta)$$

The plots of these functions are shown below





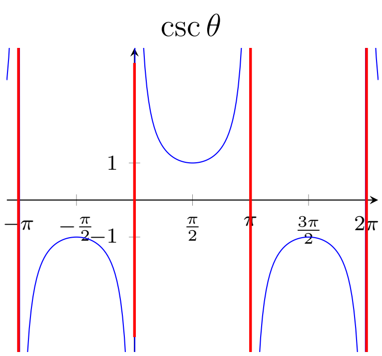


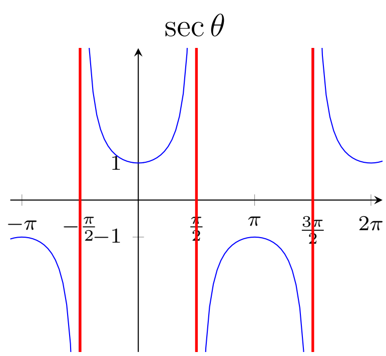
The reciprocals of these functions also play important roles in trigonometry and calculus:

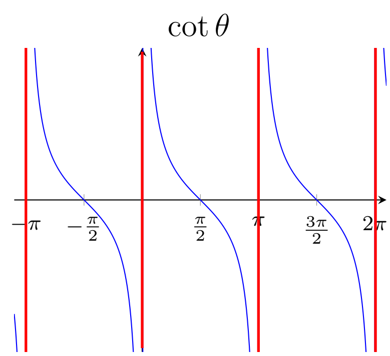
cosecant secant cotangent

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\mathrm{h}}{\mathrm{o}} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{\mathrm{h}}{\mathrm{a}} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{\mathrm{a}}{\mathrm{o}}$$

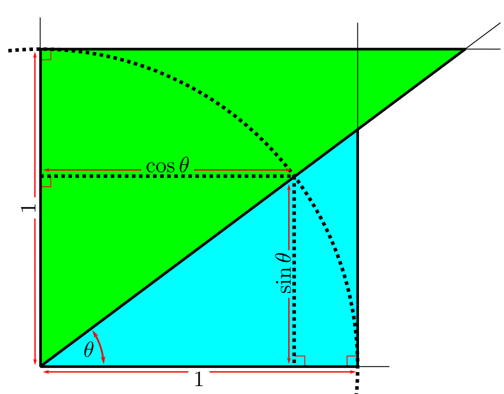
The plots of these functions are shown below

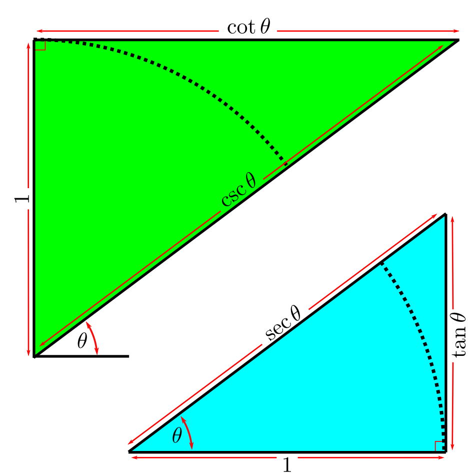






These reciprocal functions also have geometric interpretations:





Since these are all right-angled triangles we can use Pythagoras to obtain the following identities:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

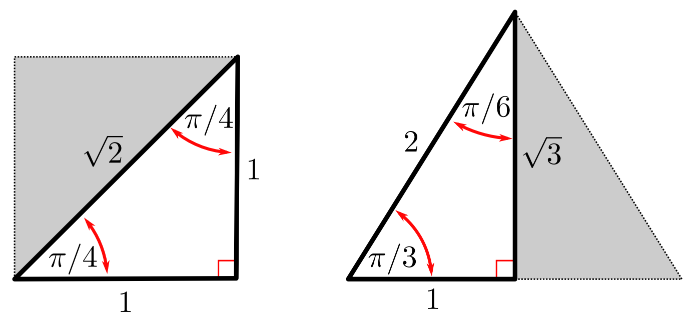
Of these it is only necessary to remember the first

$$\sin^2\theta + \cos^2\theta = 1$$

The second can then be obtained by dividing this by $\cos^2\theta$ and the third by dividing by $\sin^2\theta$.

Important triangles

Computing sine and cosine is non-trivial for general angles — we need Taylor series (or similar tools) to do this. However there are some special angles (usually small fractions of π) for which we can use a little geometry to help. Consider the following two triangles.

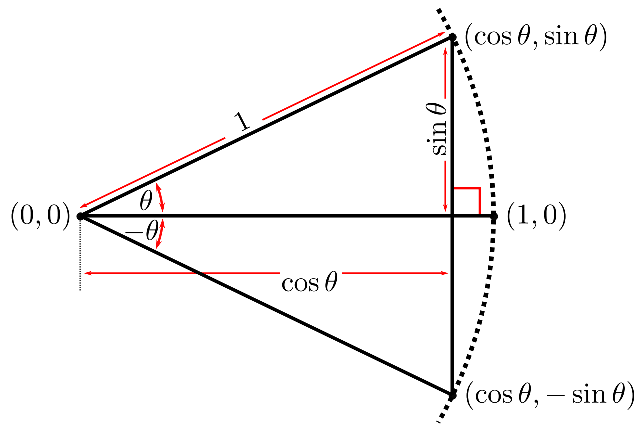


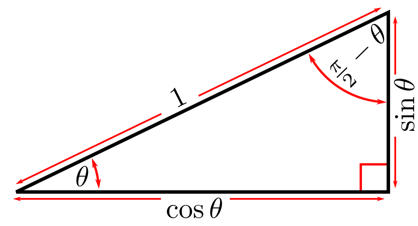
The first results from cutting a square along its diagonal, while the second is obtained by cutting an equilateral triangle from one corner to the middle of the opposite side. These, together with the angles 0, $\frac{\pi}{2}$ and π give the following table of values

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0 rad	0	1	0	DNE	1	DNE
$\frac{\pi}{2}$ rad	1	0	DNE	1	DNE	0
π rad	0	-1	0	DNE	-1	DNE
$\frac{\pi}{4}$ rad	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{6}$ rad	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$\frac{\pi}{3}$ rad	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$

Some more simple identities

Consider the figure below





The pair triangles on the left shows that there is a simple relationship between trigonometric functions evaluated at θ and at $-\theta$:

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta)$$

That is — sine is an odd function, while cosine is even. Since the other trigonometric functions can be expressed in terms of sine and cosine we obtain

$$\tan(-\theta) = -\tan(\theta) \qquad \csc(-\theta) = -\csc(\theta) \qquad \sec(-\theta) = \sec(\theta) \qquad \cot(-\theta) = -\cot(\theta)$$

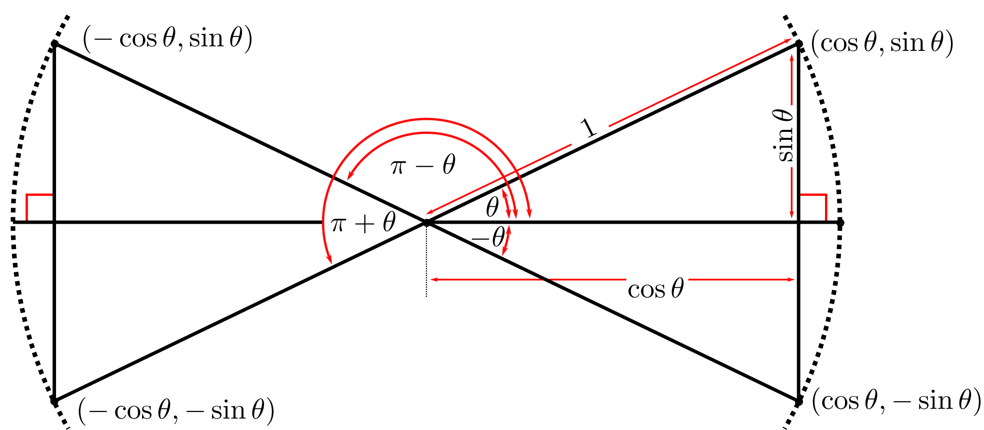
Now consider the triangle on the right — if we consider the angle $\frac{\pi}{2} - \theta$ the side-lengths of the triangle remain unchanged, but the roles of "opposite" and "adjacent" are swapped. Hence we have

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

Again these imply that

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

We can go further. Consider the following diagram:



This implies that

$$\sin(\pi - \theta) = \sin(\theta) \qquad \cos(\pi - \theta) = \cos(\theta)$$
$$\sin(\pi + \theta) = -\sin(\theta) \qquad \cos(\pi + \theta) = -\cos(\theta)$$

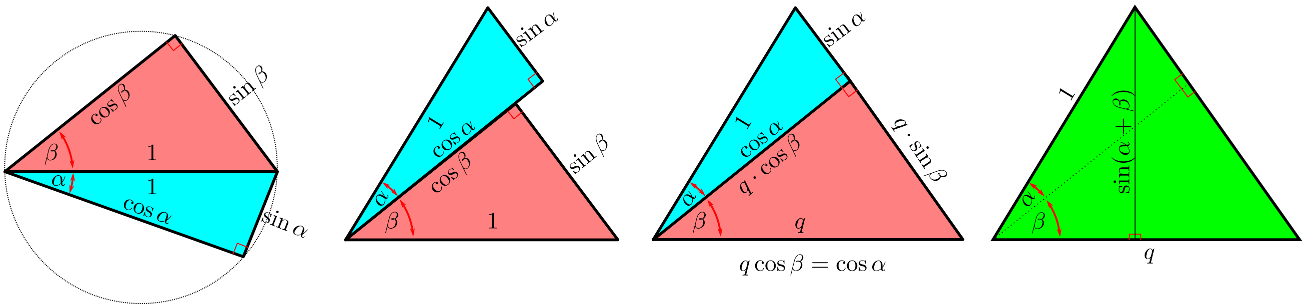
From which we can get the rules for the other four trigonometric functions.

Identities — add angles

We wish to explain the origins of the identity

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

A very geometric demonstration uses the figure below and an observation about areas.



- The left-most figure shows two right-angled triangles with angles α and β and both with hypotenuse length 1.
- The next figure simply rearranges the triangles — translating and rotating the lower triangle so that it lies adjacent to the top of the upper triangle.
- Now scale the lower triangle by a factor of q so that edges opposite the angles α and β are flush. This means that $q\cos\beta = \cos\alpha$. ie

  $$q = \frac{\cos\alpha}{\cos\beta}$$

  Now compute the areas of these (blue and red) triangles

  $$A_{\text{red}} = \frac{1}{2}q^2\sin\beta\cos\beta$$
  $$A_{\text{blue}} = \frac{1}{2}\sin\alpha\cos\alpha$$

  So twice the total area is

  $$2A_{\text{total}} = \sin\alpha\cos\alpha + q^2\sin\beta\cos\beta$$

- But we can also compute the total area using the rightmost triangle:

  $$2A_{\text{total}} = q\sin(\alpha+\beta)$$

Since the total area must be the same no matter how we compute it we have

$$\begin{aligned}
q\sin(\alpha+\beta) &= \sin\alpha\cos\alpha + q^2\sin\beta\cos\beta \\
\sin(\alpha+\beta) &= \frac{1}{q}\sin\alpha\cos\alpha + q\sin\beta\cos\beta \\
&= \frac{\cos\beta}{\cos\alpha}\sin\alpha\cos\alpha + \frac{\cos\alpha}{\cos\beta}\sin\beta\cos\beta \\
&= \sin\alpha\cos\beta + \cos\alpha\sin\beta
\end{aligned}$$

as required.

We can obtain the angle addition formula for cosine by substituting $\alpha \mapsto \pi/2-\alpha$ and $\beta \mapsto -\beta$ into our sine formula:

$$\begin{aligned} \sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \qquad \text{becomes} \\ \underbrace{\sin(\pi/2-\alpha-\beta)}_{\cos(\alpha+\beta)} &= \underbrace{\sin(\pi/2-\alpha)}_{\cos(\alpha)}\cos(-\beta) + \underbrace{\cos(\pi/2-\alpha)}_{\sin(\alpha)}\sin(-\beta) \\ \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \end{aligned}$$

where we have used $\sin(\pi/2-\theta) = \cos(\theta)$ and $\cos(\pi/2-\theta) = \sin(\theta)$.

It is then a small step to the formulas for the difference of angles. From the relation

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

we can substitute $\beta \mapsto -\beta$ and so obtain

$$\begin{aligned} \sin(\alpha-\beta) &= \sin(\alpha)\cos(-\beta) + \cos(\alpha)\sin(-\beta) \\ &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \end{aligned}$$

The formula for cosine can be obtained in a similar manner. To summarise

$$\begin{aligned} \sin(\alpha\pm\beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\ \cos(\alpha\pm\beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \end{aligned}$$

The formulas for tangent are a bit more work, but

$$\begin{aligned} \tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \\ &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \cdot \frac{\sec(\alpha)\sec(\beta)}{\sec(\alpha)\sec(\beta)} \\ &= \frac{\sin(\alpha)\sec(\alpha)+\sin(\beta)\sec(\beta)}{1-\sin(\alpha)\sec(\alpha)\sin(\beta)\sec(\beta)} \\ &= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} \end{aligned}$$

and similarly we get

$$\tan(\alpha-\beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$$

Identities — double-angle formulas

If we set $\beta = \alpha$ in the angle-addition formulas we get

$$\begin{aligned}
\sin(2\alpha) &= 2\sin(\alpha)\cos(\alpha) \\
\cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha) \\
&= 2\cos^2(\alpha) - 1 && \text{since } \sin^2\theta = 1 - \cos^2\theta \\
&= 1 - 2\sin^2(\alpha) && \text{since } \cos^2\theta = 1 - \sin^2\theta \\
\tan(2\alpha) &= \frac{2\tan(\alpha)}{1+\tan^2(\alpha)} \\
&= \frac{2}{\cot(\alpha)+\tan(\alpha)} && \text{divide top and bottom by } \tan(\alpha)
\end{aligned}$$

Identities — extras

Sums to products

Consider the identities

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \qquad \sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

If we add them together some terms on the right-hand side cancel:

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta).$$

If we now set $u = \alpha+\beta$ and $v = \alpha-\beta$ (ie $\alpha = \frac{u+v}{2}, \beta = \frac{u-v}{2}$) then

$$\sin(u) + \sin(v) = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

This transforms a sum into a product. Similarly:

$$\begin{aligned}
\sin(u) - \sin(v) &= 2\sin\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right) \\
\cos(u) + \sin(v) &= 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \\
\cos(u) - \sin(v) &= 2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)
\end{aligned}$$

Products to sums

Again consider the identities

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \qquad \sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

and add them together:

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta).$$

Then rearrange:

$$\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$$

In a similar way, start with the identities

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \qquad \cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

If we add these together we get

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

while taking their difference gives

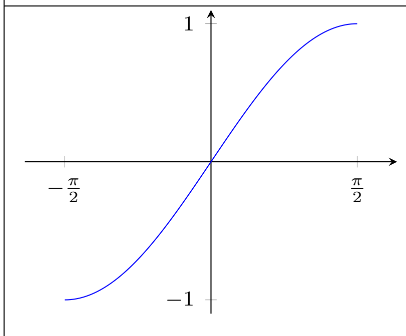
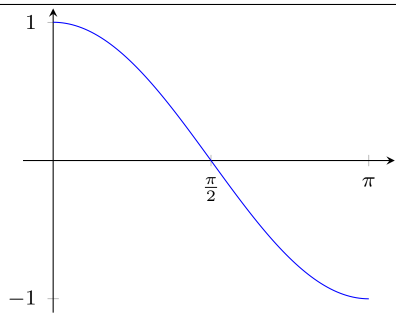
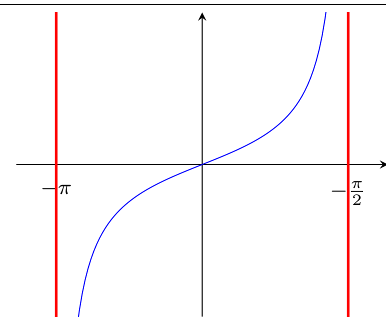
$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

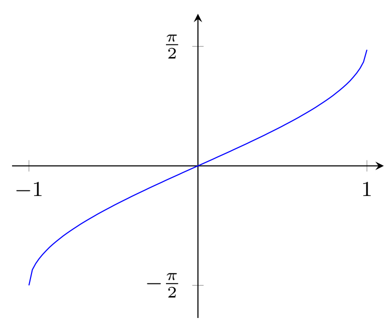
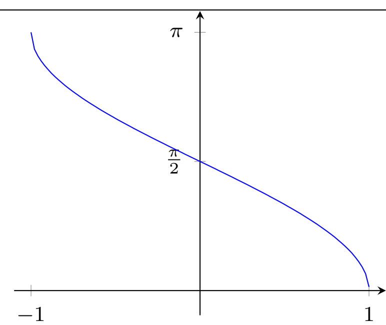
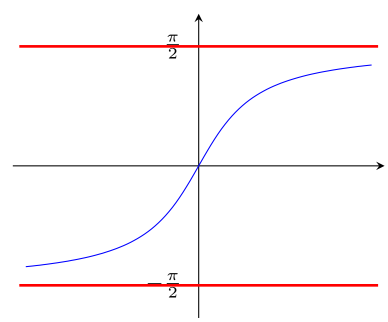
Hence

$$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$$
$$\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}$$

Inverse trigonometric functions

In order to construct inverse trigonometric functions we first have to restrict their domains so as to make them one-to-one (or injective). We do this as shown below

$\sin\theta$	$\cos\theta$	$\tan\theta$
Domain: $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ Range: $-1 \le \sin\theta \le 1$	Domain: $0 \le \theta \le \pi$ Range: $-1 \le \cos\theta \le 1$	Domain: $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ Range: all real numbers
		

$\arcsin x$	$\arccos x$	$\arctan x$
Domain: $-1 \le x \le 1$ Range: $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$	Domain: $-1 \le x \le 1$ Range: $0 \le \arccos x \le \pi$	Domain: all real numbers Range: $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$
		

Since these functions are inverses of each other we have

$$\arcsin(\sin\theta) = \theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\arccos(\cos\theta) = \theta \qquad 0 \le \theta \le \pi$$
$$\arctan(\tan\theta) = \theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

and also

$$\sin(\arcsin x) = x \qquad -1 \le x \le 1$$
$$\cos(\arccos x) = x \qquad -1 \le x \le 1$$
$$\tan(\arctan x) = x \qquad \text{any real } x$$

We can also other trig identities to determine other combinations of these functions and inverses. For example, in order to simplify $f(x) = \cos(\arcsin x)$ proceed as follows.

- First notice that we could simplify it if we can rewrite $\cos(\arcsin x)$ in terms of $\sin(\arcsin x) = x$.

- To do this we make use of one of the Pythagorean identities

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \cos\theta &= \pm\sqrt{1-\sin^2\theta} \end{aligned}$$

- Thus (setting $\theta = \arcsin x$ in the above)

$$f(x) = \pm\sqrt{1-\sin^2\theta}.$$

- To determine which branch we should use we need to consider the doman and range of $\arcsin x$:

$$\text{Domain: } -1 \le x \le 1 \qquad \text{Range: } -\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$$

  Thus we are applying cosine to an angle that always lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Cosine is non-negative on this range. Hence we should take the positive branch and

$$\begin{aligned} f(x) = \cos(\arcsin x) &= \sqrt{1-\sin^2\theta} = \sqrt{1-\sin^2(\arcsin x)} \\ &= \sqrt{1-x^2} \end{aligned}$$

In a very similar way we can simplify $\tan\arccos x$.

- First notice that we could simplify it if we can rewrite $\tan(\arccos x)$ in terms of $\cos(\arccos x) = x$.
- To do this we make use of one of the Pythagorean identities

$$\begin{aligned} \tan^2\theta + 1 = \sec^2\theta &= \frac{1}{\cos^2\theta} \\ \tan\theta &= \pm\sqrt{1/\cos^2\theta - 1} \end{aligned}$$

- Thus (setting $\theta = \arccos x$ in the above)

$$\tan\arccos x = \pm\sqrt{1/\cos^2(\arccos x) - 1}.$$

- To determine which branch we should use we need to consider the doman and range of $\arccos x$:

$$\text{Domain: } -1 \le x \le 1 \qquad \text{Range: } 0 \le \arccos x \le \pi$$

  Thus we are applying cosine to an angle that always lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Cosine is non-negative on this range. Hence we should take the positive branch and

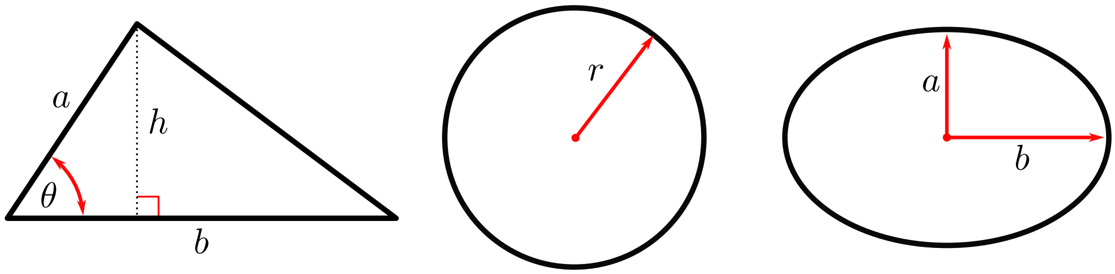
$$\begin{aligned} f(x) = \cos(\arcsin x) &= \sqrt{1-\sin^2\theta} = \sqrt{1-\sin^2(\arcsin x)} \\ &= \sqrt{1-x^2} \end{aligned}$$

Completing the 9 possibilities gives:

$$\begin{array}{lll} \sin(\arcsin x) = x & \sin(\arccos x) = \sqrt{1-x^2} & \sin(\arctan x) = \dfrac{x}{\sqrt{1+x^2}} \\ \cos(\arcsin x) = \sqrt{1-x^2} & \cos(\arccos x) = x & \cos(\arctan x) = \dfrac{1}{\sqrt{1+x^2}} \\ \tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}} & \tan(\arccos x) = \dfrac{\sqrt{1-x^2}}{x} & \tan(\arctan x) = x \end{array}$$

Geometry

Areas



- Area of a triangle

$$A = \frac{1}{2}bh = \frac{1}{2}ab\sin\theta$$

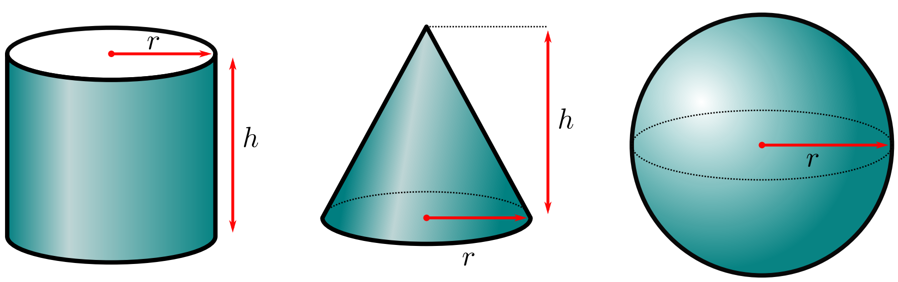
- Area of a circle

$$A = \pi r^2$$

- Area of an elipse

$$A = \pi ab$$

Volumes



- Volume of a cylinder

$$V = \pi r^2 h$$

- Volume of a cone

$$V = \frac{1}{3}\pi r^2 h$$

- Volume of a sphere

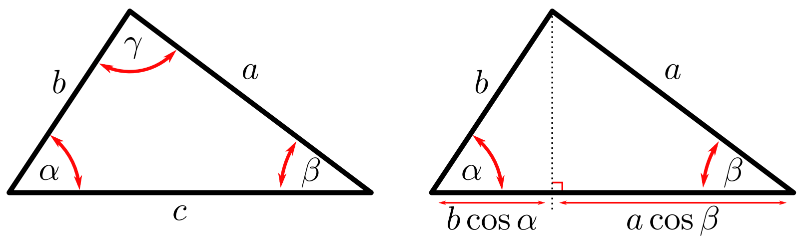
$$V = \frac{4}{3}\pi r^3$$

Where do the sine and cosine laws for triangles come from

We do not actually need you to remember these, but we can explain how they arise.

Cosine law or Law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$



Consider the triangle on the left. Now draw a perpendicular line from the side of length c to the opposite corner as shown. This demonstrates that

$$c = a\cos\beta + b\cos\alpha$$

Multiply this by c to get an expression for c^2:

$$c^2 = ac\cos\beta + bc\cos\alpha$$

Doing similarly for the other corners gives

$$a^2 = ac\cos\beta + ab\cos\gamma$$
$$b^2 = bc\cos\alpha + ab\cos\gamma$$

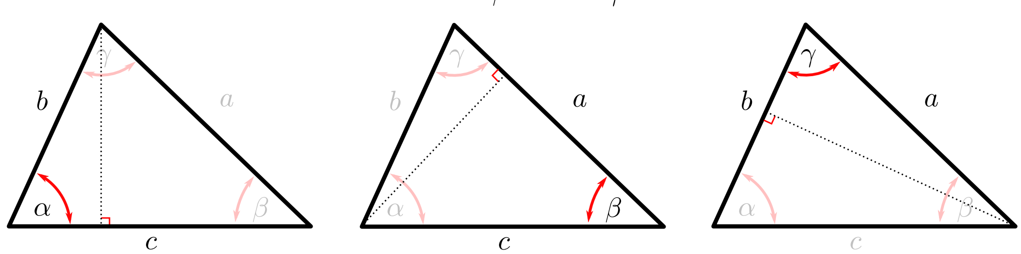
Now combining these:

$$\begin{aligned} a^2 + b^2 - c^2 &= (bc - bc)\cos\alpha + (ac - ac)\cos\beta + 2ab\cos\gamma \\ &= 2ab\cos\gamma \end{aligned}$$

as required.

Sine law or Law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = R$$



This rule is best understood by computing the area of the triangle using the formula $A = \frac{1}{2}ab\sin\theta$ seen above. Doing this three ways gives

$$2A = bc\sin\alpha$$
$$2A = ac\sin\beta$$
$$2A = ab\sin\gamma$$

Dividing these expressions by abc gives

$$\frac{2A}{abc} = \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

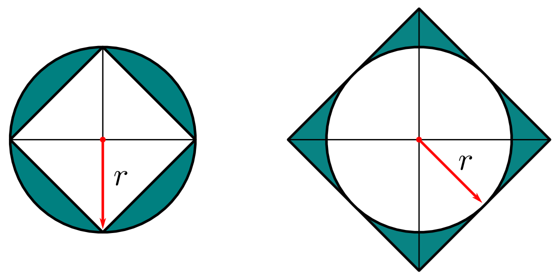
as required.

Where does the formula for the area of a circle come from

Typically when we come across π for the first time it is as the ratio of the circumference of a circle to its diameter

$$\pi = \frac{C}{d} = \frac{C}{2r}$$

Indeed this is typically the first definition we see of π. It is easy to build an intuition that the area of the circle should be propotional to the square of its radius. For example we can draw the largest possible square inside the circle (an *inscribed* square) and the smallest possible square outside the circle (a *circumscribed* square):



The smaller square has side-length $\sqrt{2}r$ and the longer has side-length $2r$. Hence

$$2r^2 \leq A \leq 4r^2 \qquad \text{or } 2 \leq \frac{A}{r^2} \leq 4$$

That is, the area of the circle is between 2 and 4 times the square of the radius. What is perhaps less obvious (if we had not been told this in school) is that the constant of propotionality for area is also π:

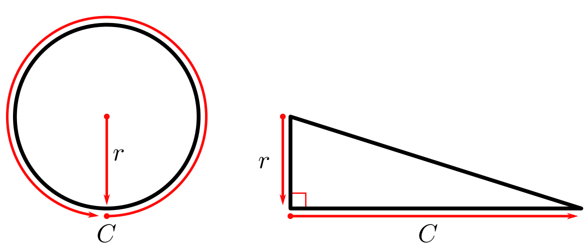
$$\pi = \frac{A}{r^2}.$$

We will show this using Archimedes' proof. He makes use of these inscribed and circumscribed polygons to make better and better approximations of the circle. The steps of the proof are somewhat involved and the starting point is to rewrite the area of a circle as

$$A = \frac{1}{2}Cr$$

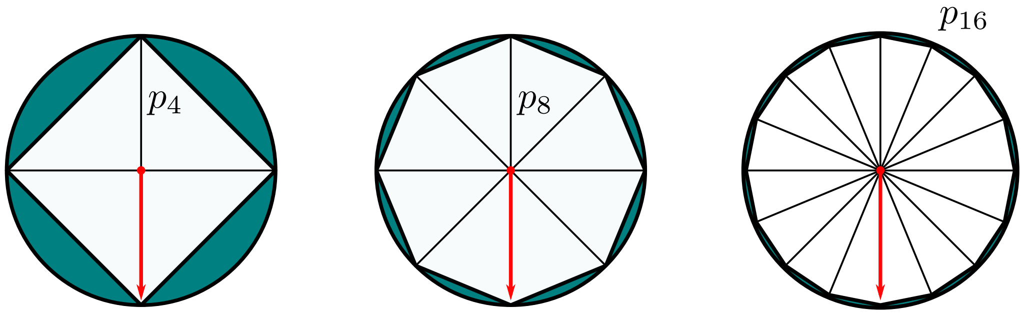
This suggests that this area is the same as that of a triangle of height r and base length C

$$T = \frac{1}{2}Cr$$

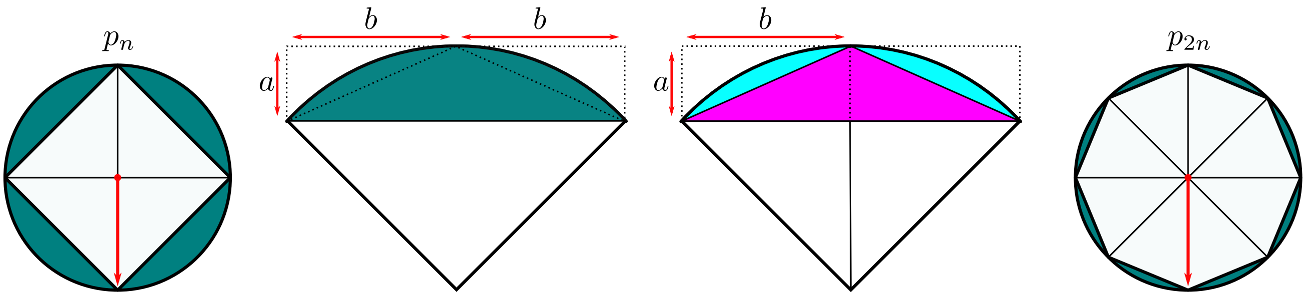


Archimedes' proof then demonstrates that indeed this triangle and the circle have the same area. It relies on a "proof by contradiction" — showing that $T < A$ and $T > A$ cannot be true and so the only possibility is that $A = T$.

We will first show that $T < A$ cannot happen. Construct an n-sided "inscribed" polygon as shown below:



Let p_n be the inscribed polygon as shown.



We need 4 steps.

1. The area of p_n is smaller than that of the circle — this follows since we can construct p_n by cutting slices from the circle.

2. Let E_n be the difference between the area of the circle and p_n: $E_n = A - A(p_n)$ (see the left of the previous figure). By the previous point we know $E_n > 0$. Now as we increase the number of sides, this difference becomes smaller. To be more precise

$$E_{2n} \leq \frac{1}{2} E_n.$$

The error E_n is made up of n "lobes". In the centre-left of the previous figure we draw one such lobe and surround it by a rectangle of dimensions $a \times 2b$ — we could determine these more precisely using a little trigonometry, but it is not necessary.

This diagram shows the lobe is smaller than the rectangle of base $2b$ and height a Since there are n copies of the lobe, we have

$$E_n \leq n \times 2ab \qquad \text{rewrite as } \frac{E_n}{2} \leq nab$$

Now draw in the polygon p_{2n} and consider the associated "error" E_{2n}. If we focus on the two lobes shown then we see that the area of these two new lobes is equal to that of the old lobe (shown in centre-left) minus the area of the triangle with base $2b$ and height a (drawn in purple). Since there are n copies of this picture we have

$$\begin{aligned} E_{2n} &= E_n - nab & &\text{now use that } nab > E_n/2 \\ &< E_n - \frac{E_n}{2} = \frac{E_n}{2} \end{aligned}$$

3. The area of p_n is smaller than T. To see this decompose p_n into n isosceles triangles. Each of these has base shorter than C/n; the straight line is shorter than the corresponding arc — though strictly speaking we should prove this. The height of each triangle is shorter than r. Thus

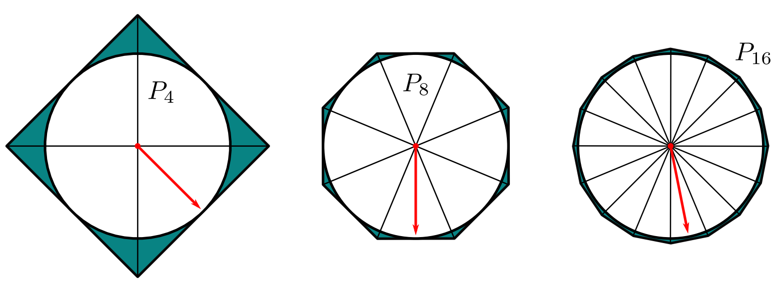
$$\begin{aligned} A(p_n) &= n \times \frac{1}{2}(\text{base}) \times (\text{height}) \\ &\leq n \times \frac{Cr}{2n} = T \end{aligned}$$

4. If we assume that $A < T$, then $A - T = d$ where d is some positive number. However we know from point 2 that we can make n large enough so that $E_n < d$ (each time we double n we halve the error). But now we have a contradiction since we have just shown that

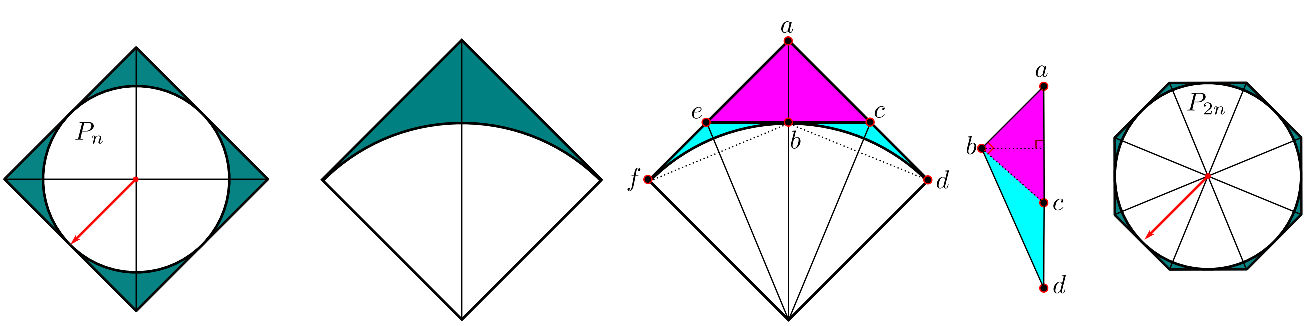
$$\begin{aligned} E_n = A - A(p_n) &\leq A - T & &\text{which implies that} \\ A(p_n) &\geq T. \end{aligned}$$

Thus we cannot have $T < A$.

If we now assume that $T > A$ we will get a similar contradiction by a similar construction. Now we use regular n-sided *circumscribed* polygons, P_n.



The proof can be broken into 4 similar steps.



1. The area of P_n is greater than that of the circle — this follows since we can construct the circle by trimming the polygon P_n.

2. Let E_n be the difference between the area of the polygon and the circle: $E_n = A(P_n) - A$ (see the left of the previous figure). By the previous point we know $E_n > 0$. Now as we increase the number of sides, this difference becomes smaller. To be more precise we will show

$$E_{2n} \leq \frac{1}{2}E_n.$$

The error E_n is made up of n "lobes". In the centre-left of the previous figure we draw one such lobe. Let L_n denote the area of one of these lobes, so $E_n = nL_n$. In the centre of the previous figure we have labelled this lobe carefully and also shown how it changes when we create the polygon P_{2n}. In particular, the original lobe is bounded by the straight lines $\vec{ad}, \vec{af}$ and the arc $\widehat{fbd}$. We create P_{2n} from P_n by cutting away the corner triangle $\triangle aec$. Accordingly the lines $\vec{ec}$ and $\vec{ba}$ are orthogonal and the segments $|bc| = |cd|$.

By the construction of P_{2n} from P_n, we have

$$2L_{2n} = L_n - A(\triangle aec) \qquad \text{or equivalently } L_{2n} = \frac{1}{2}L_n - A(\triangle abc)$$

And additionally

$$L_{2n} \leq A(\triangle bcd)$$

Now consider the triangle $\triangle abd$ (centre-right of the previous figure) and the two triangles within it $\triangle abc$ and $\triangle bcd$. We know that $\vec{ab}$ and $\vec{cb}$ form a right-angle. Consequently $\vec{ac}$ is the hypotenuse of a right-angled triangle, so $|ac| > |bc| = |cd|$. So now, the triangles $\triangle abc$ and $\triangle bcd$ have the same heights, but the base of $\vec{ac}$ is longer than $\vec{cd}$. Hence the area of $\triangle abc$ is strictly larger than that of $\triangle bcd$.

Thus we have

$$L_{2n} \leq A(\triangle bcd) < A(\triangle abc)$$

But now we can write

$$\begin{aligned}
L_{2n} &= \frac{1}{2}L_n - A(\triangle abc) < \frac{1}{2}L_n - L_{2n} && \text{rearrange} \\
2L_{2n} &< \frac{1}{2}L_n && \text{there are } n \text{ such lobes, so} \\
2nL_{2n} &< \frac{n}{2}L_n && \text{since } E_n = nL_n \text{, we have} \\
E_{2n} &< \frac{1}{2}E_n && \text{which is what we wanted to show.}
\end{aligned}$$

3. The area of P_n is greater than T. To see this decompose P_n into n isosceles triangles. The height of each triangle is r, while the base of each is longer than C/n (this is a subtle point and its proof is equivalent to showing that $\tan\theta > \theta$). Thus

$$\begin{aligned}
A(P_n) &= n \times \frac{1}{2}(\text{base}) \times (\text{height}) \\
&\geq n \times \frac{Cr}{2n} = T
\end{aligned}$$

4. If we assume that $T > A$, then $T - A = d$ where d is some positive number. However we know from point 2 that we can make n large enough so that $E_n < d$ (each time we double n we halve the error). But now we have a contradiction since we have just shown that

$$E_n = A(P_n) - A \le T - A \qquad \text{which implies that}$$
$$A(p_n) \le T.$$

Thus we cannot have $T > A$. The only possibility that remains is that $T = A$.

Where do these volume formulas come from

We can establish the volumes of cones and spheres from the formula for the volume of a cylinder and a little work with limits and some careful summations. We first need a few facts.

- Every square number can be written as a sum of consecutive odd numbers. More precisely

$$n^2 = 1 + 3 + \dots (2n - 1)$$

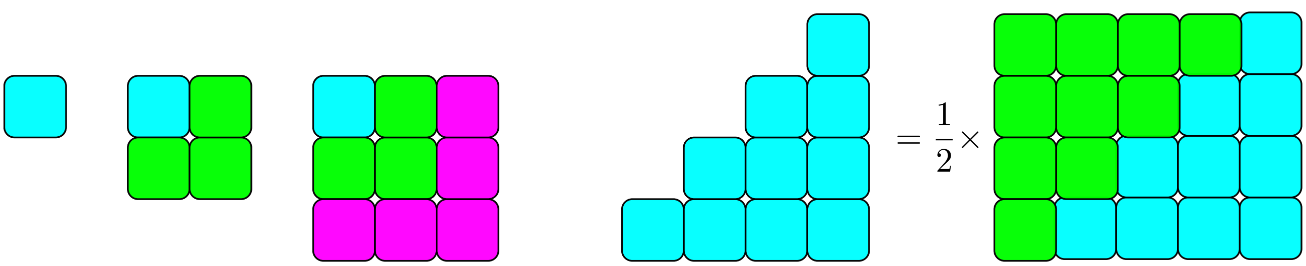
- The sum of the first n positive integers is $\frac{1}{2}n(n+1)$. That is

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$$

- The sum of the squares of the first n positive integers is $\frac{1}{6}n(n+1)(2n+1)$.

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

We will not give completely rigorous proofs of the above identities (since we are not going to assume that the reader knows mathematical induction), rather we will explain them using pictorial arguments. The first two of these we can explain by some quite simple pictures:

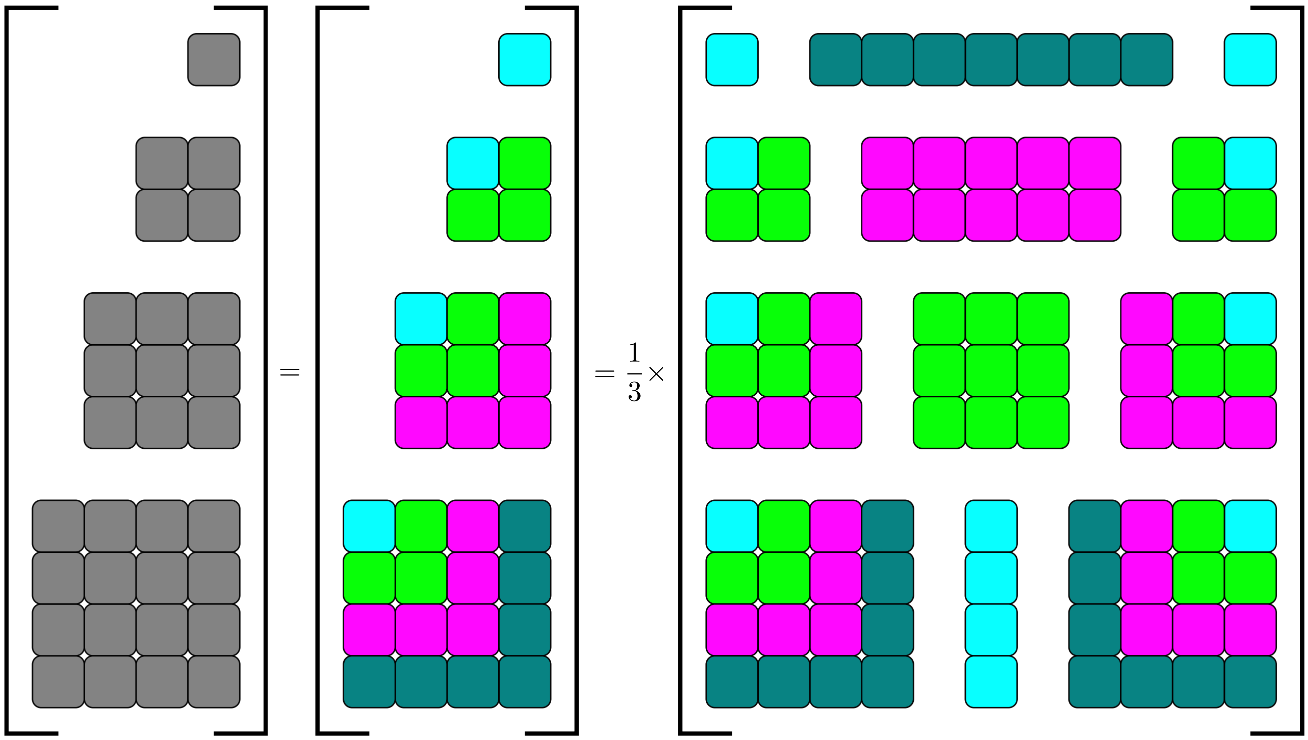


We see that we can decompose any square of unit-squares into a sequence of strips, each of which consists of an odd number of unit-squares. This is really just from the fact that

$$n^2 - (n-1)^2 = 2n - 1$$

Similarly, we can represent the sum of the first n integers as a triangle of unit squares as shown. If we make a second copy of that triangle and arrange it as shown, it gives a rectangle of dimensions n by $n + 1$. Hence the rectangle, being exactly twice the size of the original triangle, contains $n(n+1)$ unit squares.

The explanation of the last formula takes a little more work and a carefully constructed picture:



Let us break these pictures down step by step

- Leftmost represents the sum of the squares of the first n integers.
- Centre — We recall from above that each square number can be written as a sum of consecutive odd numbers, which have been represented as coloured bands of unit-squares.
- Make three copies of the sum and arrange them carefully as shown. The first and third copies are obvious, but the central copy is rearranged considerably; all the bands of the same lengths have been arranged them into rectangles as shown.

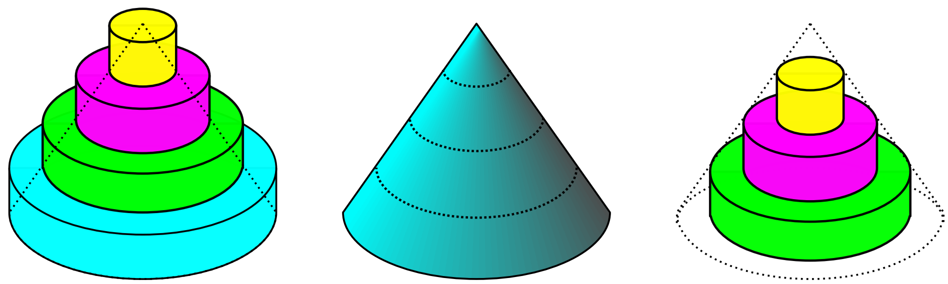
  Putting everything from the three copies together creates a rectangle of dimensions $(2n+1) \times (1+2+3+\cdots+n)$.

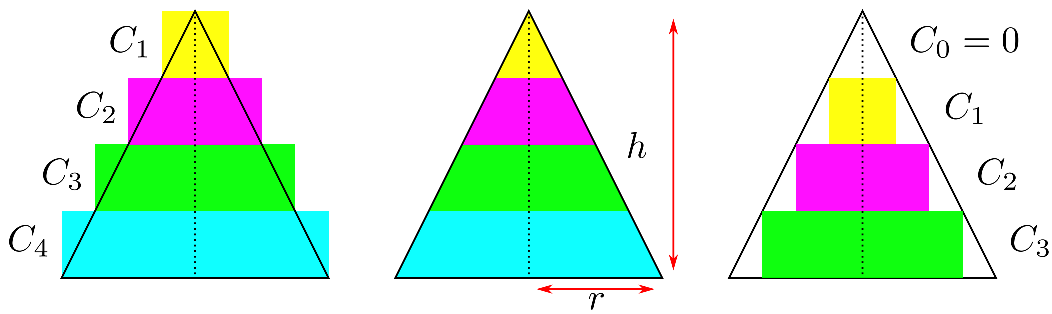
We know (from above) that $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$ and so

$$(1^2+2^2+\cdots+n^2) = \frac{1}{3} \times \frac{1}{2}n(n+1)(2n+1)$$

as required.

Now we can start to look at volumes. Let us start with the volume of a cone; consider the figure below. We bound the volume of the cone above and below by stacks of cylinders. The cross-sections of the cylinders and cone are also shown.





To obtain the bounds we will construct two stacks of n cylinders, $C_1, C_2, \ldots, C_n$. Each cylinder has height h/n and radius that varies with height. In particular, we define cylinder C_k to have height h/n and radius $k \times r/n$. This radius was determined using similar triangles so that cylinder C_n has radius r. Now cylinder C_k has volume

$$\begin{aligned} V_k &= \pi \times \text{radius}^2 \times \text{height} = \pi \left(\frac{kr}{n}\right)^2 \cdot \frac{h}{n} \\ &= \frac{\pi r^2 h}{n^3} k^2 \end{aligned}$$

We obtain an upper bound by stacking cylinders $C_1, C_2, \ldots, C_n$ as shown. This object has volume

$$\begin{aligned} V &= V_1 + V_2 + \ldots V_n \\ &= \frac{\pi r^2 h}{n^3} \left(1^2 + 2^2 + 3^2 + \cdots + n^2\right) \\ &= \frac{\pi r^2 h}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

A similar lower bound is obtained by stacking cylinders $C_1, \ldots, C_{n-1}$ which gives a volume of

$$\begin{aligned} V &= V_1 + V_2 + \ldots V_{n-1} \\ &= \frac{\pi r^2 h}{n^3} \left(1^2 + 2^2 + 3^2 + \cdots + (n-1)^2\right) \\ &= \frac{\pi r^2 h}{n^3} \cdot \frac{(n-1)(n)(2n-1)}{6} \end{aligned}$$

Thus the true volume of the cylinder is bounded between

$$\frac{\pi r^2 h}{n^3} \cdot \frac{(n-1)(n)(2n-1)}{6} \leq \text{correct volume} \leq \frac{\pi r^2 h}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

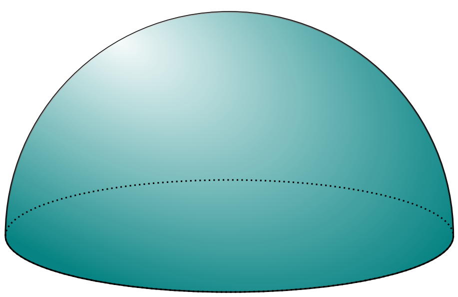
We can now take the limit as the number of cylinders, n, goes to infinity. The upper bound becomes

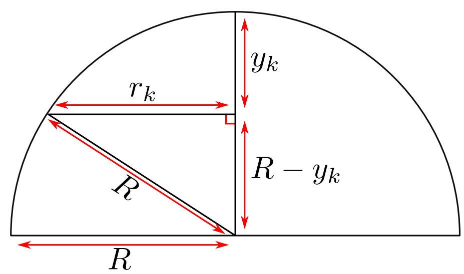
$$\begin{aligned}
\lim_{n\to\infty} \frac{\pi r^2 h}{n^3} \frac{n(n+1)(2n+1)}{6} &= \frac{\pi r^2 h}{6} \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{n^3} \\
&= \frac{\pi r^2 h}{6} \lim_{n\to\infty} \frac{(1+1/n)(2+1/n)}{1} \\
&= \frac{\pi r^2 h}{6} \times 2 \\
&= \frac{\pi r^2 h}{3}
\end{aligned}$$

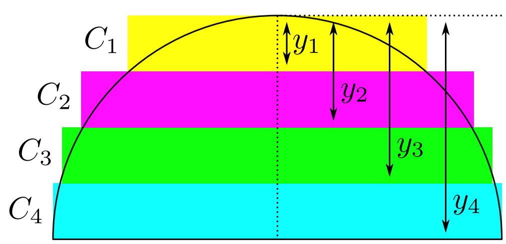
The other limit is identical, so by the squeeze theorem we have

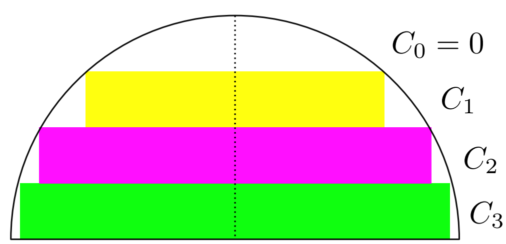
$$\text{Volume of cone } = \frac{1}{3}\pi r^2 h$$

Now the sphere — though we will do the analysis for a hemisphere of radius R. Again we bound the volume above and below by stacks of cylinders. The cross-sections of the cylinders and cone are also shown.









To obtain the bounds we will construct two stacks of n cylinders, $C_1, C_2, \ldots, C_n$. Each cylinder has height h/n and radius that varies with its position in the stack. To describe the position, define

$$y_k = k \times \frac{h}{n}$$

That is, y_k, is k steps of distance $\frac{h}{n}$ from the top of the hemisphere. Then we set the k^{th} cylinder, C_k to have height R/n and radius r_k given by

$$\begin{aligned}
r_k^2 &= R^2 - (R - y_k)^2 = R^2 - R^2(1 - k/n)^2 \\
&= R^2(2k/n - k^2/n^2)
\end{aligned}$$

as shown in the top-right and bottom-left illustrations. The volume of C_k is then

$$\begin{aligned} V_k &= \pi \times \text{radius}^2 \times \text{height} = \pi \times R^2 \left(2k/n - k^2/n^2\right) \times \frac{R}{n} \\ &= \pi R^3 \cdot \left(\frac{2k}{n^2} - \frac{k^2}{n^3}\right) \end{aligned}$$

We obtain an upper bound by stacking cylinders $C_1, C_2, \ldots, C_n$ as shown. This object has volume

$$\begin{aligned} V &= V_1 + V_2 + \ldots V_n \\ &= \pi R^3 \cdot \left(\frac{2}{n^2}\left(1+2+3+\cdots+n\right) - \frac{1}{n^3}\left(1^2+2^2+3^2+\cdots+n^2\right)\right) \end{aligned}$$

Now recall from above that

$$1+2+3+\cdots+n = \frac{1}{2}n(n+1) \qquad 1^2+2^2+3^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$$

so

$$V = \pi R^3 \cdot \left(\frac{n(n+1)}{n^2} - \frac{n(n+1)(2n+1)}{6n^3}\right)$$

Again, a lower bound is obtained by stacking cylinders $C_1, \ldots, C_{n-1}$ and a similar analysis gives

$$V = \pi R^3 \cdot \left(\frac{n(n-1)}{(n-1)^2} - \frac{n(n-1)(2n-1)}{6(n-1)^3}\right)$$

Thus the true volume of the hemisphere is bounded between

$$\pi R^3 \cdot \left(\frac{n(n+1)}{n^2} - \frac{n(n+1)(2n+1)}{6n^3}\right) \leq \text{correct volume} \leq \pi R^3 \cdot \left(\frac{n(n+1)}{n^2} - \frac{n(n+1)(2n+1)}{6n^3}\right)$$

We can now take the limit as the number of cylinders, n, goes to infinity. The upper bound becomes

$$\begin{aligned} \lim_{n\to\infty} \pi R^3 \cdot \left(\frac{n(n+1)}{n^2} - \frac{n(n+1)(2n+1)}{6n^3}\right) &= \pi R^2 \left(\lim_{n\to\infty} \frac{n(n+1)}{n^2} - \frac{n(n+1)(2n+1)}{6n^3}\right) \\ &= \pi R^3 \left(1 - \frac{2}{6}\right) = \frac{2}{3}\pi R^3. \end{aligned}$$

The other limit is identical, so by the squeeze theorem we have

$$\begin{aligned} \text{Volume of hemisphere } &= \frac{2}{3}\pi R^3 \qquad \text{and so} \\ \text{Volume of sphere } &= \frac{4}{3}\pi R^3 \end{aligned}$$