Math 190 Homework 10: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:

1. Compute the following integrals

   (a) $\int \frac{\cos x}{\sin x}\, dx$

   (b) $\int e^x \sin(2+e^x)\, dx$

   Solution: (a) Let $u = \sin x$. Note that the denominator is a good guess. Also note that $\sin x$ is the derivative of $\cos x$. Now, we have $du = \cos x dx$ and so

   $$\int \frac{\cos x}{\sin x}\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln|\sin x| + C$$

   after switching back to x.

   (b) Let $u = 2+e^x$. Note that $2+e^2$ is the interior part of one function and also that e^x is the derivative of $2+e^x$. In this way we have $du = e^x dx$ and so

   $$\int e^x \sin(2+e^x)\ dx = \int \sin u\, du = -\cos u + C = -\cos(2+e^x) + C.$$

2. Compute the following integral

   $$\int \left(x^2-2\right)\sqrt{x+1}\, dx.$$

   (If you're lacking inspiration see the note from November 16.)

   Solution: Let $u = x+1$ the portion under the square root. In this way we simplify the root. Then we have $du = dx$ and so

   $$\int \left(x^2-2\right)\sqrt{x+1}\, dx = \int (x^2+2)\sqrt{u}\, du.$$

   We still have a problem though since we have not eliminated all the x's. However, if we observe our relationship between u and x we find that $x = u-1$ or rather $x^2 = (u-1)^2$. Substituting this to our integral yields

$$\begin{aligned}
\int \left(x^2 - 2\right) \sqrt{x+1}\, dx &= \int (x^2 + 2)\sqrt{u}\, du \\
&= \int \left((u-1)^2 - 2\right) \sqrt{u}\, du \\
&= \int (u^2 - 2u + 1 - 2)\sqrt{u}\, du \\
&= \int (u^2 - 2u - 1)u^{1/2}\, du \\
&= \int \left(u^{5/2} - 2u^{3/2} - u^{1/2}\right)\, du \\
&= \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C \\
&= \frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.
\end{aligned}$$

Note that once we had the integral only in terms of u we only needed to expand and integrate.

3. A function is called *odd* if

$$f(-x) = -f(x)$$

for all values of x.

(a) Using a picture, explain why you suspect that

$$\int_{-a}^{a} f(x)dx = 0$$

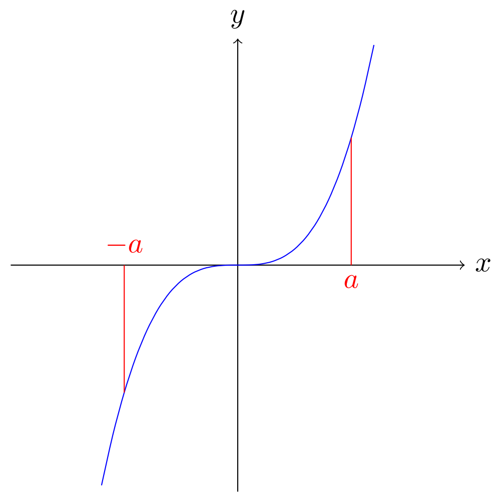
(b) Prove using a substitution that

$$\int_{-a}^{a} f(x)dx = 0$$

for any odd function $f(x)$ and any value of a.

Solution:

(a) Here is the graph of an odd function:



Observe that the magnitude of the area under the curve from $-a$ to 0 is the same as the area under the curve from 0 to a. In this way we expect the negative area to cancel exactly with the positive area. That is

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = 0.$$

(b) We will now prove this fact using a substitution. We first split the integral as follows

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

and then make the substitution $u = -x$ for the first integral only. In this way $du = -dx$ and so (working with just the first integral)

$$\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-u)du.$$

Note that we changed the bounds of integration. That is when $x = -a$ we have $u = a$ and when $x = 0$ we have $u = 0$. Recall that $f(x)$ was defined to be odd. That is to say that $f(-u) = -f(u)$. Substituting this yields

$$\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-u)du = \int_{a}^{0} f(u)du = -\int_{0}^{a} f(u)du$$

where in the last step we have reversed the bounds at the cost of a minus sign. Note also that

$$-\int_{0}^{a} f(u)du = -\int_{0}^{a} f(x)dx$$

since we are just changing the name of our variable. Therefore, after putting everything back together we see

$$\int_{-a}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0$$

as required.

If you're having trouble with the following fact

$$\int_0^a f(u)du = \int_0^a f(x)dx$$

imagine we had the anti-derivative F. Let us investigate each side

$$\int_0^a f(x)dx = F(x)\Big|_0^a = F(a) - F(0)$$

and

$$\int_0^a f(u)du = F(u)\Big|_0^a = F(a) - F(0).$$

After applying the fundamental theorem of calculus we see that each side is the same. It doesn't matter whether the variable is u or x, at the end of the day we are just going to sub in the bounds anyway. Sometimes we call the variable inside the integral a 'dummy variable.'

4. Compute the following definite integral

$$\int_{-\pi}^{\pi} x\cos\left(x^2\right)dx.$$

Explain, in reference to Question 3, why you expected this result.

Solution: We proceed with a substitution. Let $u = x^2$. We note that x^2 is the inside part of one function but also that the derivative of x^2 is 'kind of like' x. We proceed noting that $du = 2xdx$. Observe

$$\int_{x=-\pi}^{x=\pi} x\cos\left(x^2\right)\,dx = \frac{1}{2}\int_{u=\pi^2}^{u=\pi^2} \cos(u)\,du$$

where we have changed the bounds of integration. That is, when $x = -\pi$ we have that $u = (-\pi)^2 = \pi^2$. Similarly when $x = \pi$ we have $u = \pi^2$. We could stop at this stage and declare the integral zero since both bounds are the same; we are sweeping out no area. Alternatively we can compute the integral, but it is zero non the less

$$\frac{1}{2}\int_{u=\pi^2}^{u=\pi^2} \cos(u)\,du = \frac{1}{2}\sin(u)\Big|_{\pi^2}^{\pi^2} = \frac{1}{2}\left(\sin(\pi^2) - \sin(\pi^2)\right) = 0.$$

From the onset we would expect this integral to be zero based on our insights from problem 3. We are integrating an odd function over symmetric limits and so the value of the integral will be zero. We will demonstrate now that our integrand is odd. Let $f(x) = x\cos(x^2)$ and consider

$$f(-x) = (-x)\cos\left((-x)^2\right) = -x\cos(x^2) = -f(x).$$

Since $f(-x) = -f(x)$ we have established the claim that our integrand is odd.

5. Evaluate the following indefinite integral

$$\int \frac{\sin x\,(\cos x + 1)}{\cos x}\,dx.$$

Solution: Let us try a substitution. Consider $u = \cos x$. This is a good choice because it will allow us to simplify the denominator as well as get rid of the $\sin x$ which is 'kind of like' the derivative of $\cos x$. Note that $du = -\sin x\,dx$ and so we see

$$\begin{aligned}\int \frac{(\cos x + 1)}{\cos x}\sin x\,dx &= -\int \frac{u+1}{u}\,du \\ &= -\int \left(1 + \frac{1}{u}\right)\,du \\ &= -u - \ln|u| + C \\ &= -\cos x - \ln|\cos x| + C.\end{aligned}$$

Done!