Math 190 Homework 1: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:

1. Find all $x \in \mathbb{R}$ such that

$$x^4 - 2x^2 - 1 = 0.$$

Solution: Solving a general quartic equation is a difficult task. We notice, however, that this special quartic is lacking odd powers of x. We can take advantage of this nicety by making the substitution $u = x^2$. In light of this our equation is transformed to

$$u^2 - 2u - 1 = 0.$$

We can solve the above quadratic equation by means of the quadratic formula. That is

$$\begin{aligned} u =& \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} \\ =& \frac{2 \pm \sqrt{8}}{2} \\ =& \frac{2 \pm 2\sqrt{2}}{2} \\ =& 1 \pm \sqrt{2}. \end{aligned}$$

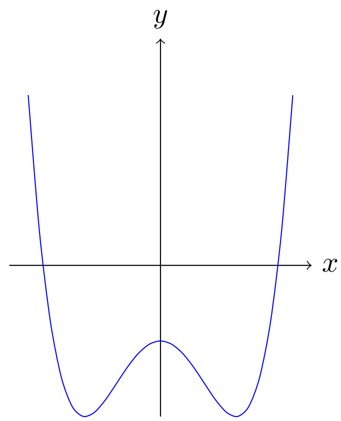
Having successfully found u we turn our attention to finding x. We would like to write

$$x = \pm\sqrt{u} = \pm\sqrt{1 \pm \sqrt{2}}$$

however we are only interested in *real* values for x. Observe that $\sqrt{2} > 1$ and so $1 - \sqrt{2} < 0$. Since there is no real number which is the square root of a negative number we must eliminate two of our four above solutions. Therefore two our values for x are

$$x = \pm\sqrt{1 + \sqrt{2}}.$$

A quick sketch of the graph (not required) reveals that indeed to do expect two solutions:



2. Find all (real) zeros of

$$g(x) = \begin{cases} x^2 - 9, & x \leq 0 \\ -\frac{1}{3}(x-5)+1, & x > 0 \end{cases}.$$

Consider sketching the graph of g.

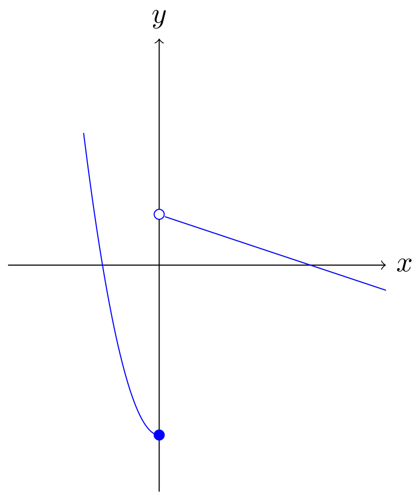
Solution: Let us find the zeros of each branch. First we solve

$$\begin{aligned} x^2 - 9 &= 0 \\ (x-3)(x+3) &= 0 \end{aligned}$$

which yields $x = -3$ and $x = 3$. It is true that $g(-3) = 0$ since $-3 < 0$. We do not, however, have $g(3) = 0$ since for $x > 0$ we would select the second branch. Now we solve

$$\begin{aligned} -\frac{1}{3}(x-5)+1 &= 0 \\ -\frac{1}{3}(x-5) &= -1 \\ x - 5 &= 3 \\ x &= 8. \end{aligned}$$

Since $8 > 0$ we do in fact have $g(8) = 0$. All together our two zeros are $x = -3$ and $x = 8$. A quick sketch of the graph (not required) confirms that we should be expecting two zeros:

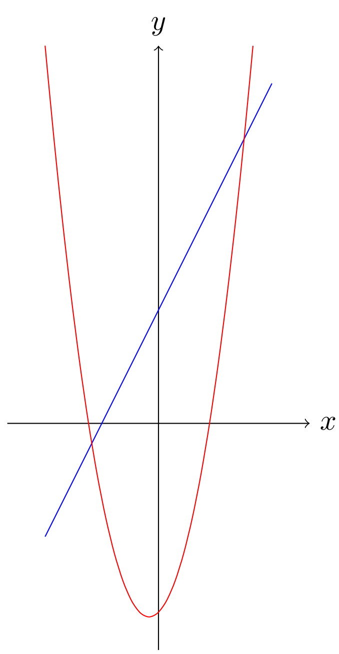


3. Find all real t such that the function

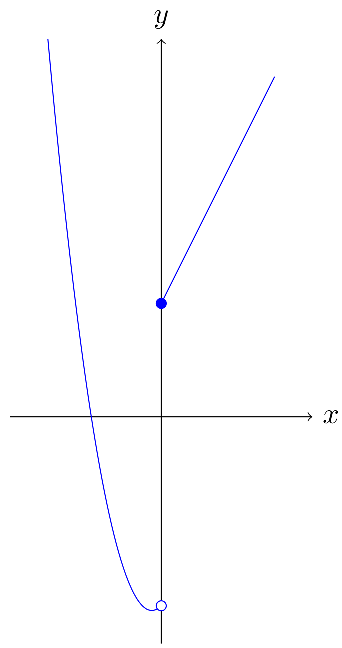
$$f(x) = \begin{cases} 2x+3, & x \geq t \\ 2x^2+x-5, & x < t \end{cases}$$

is continuous. Support your answer with one or more pictures.

Solution: To ensure that f is continuous we want both branches to line up at $x = t$. If we plot both branches for all values of x we see that we expect two acceptable values of t.



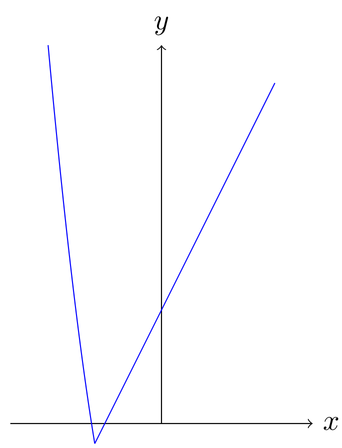
To further emphasize this point (not required) we can observe that taking $t = 0$ will give a graph that is not continuous

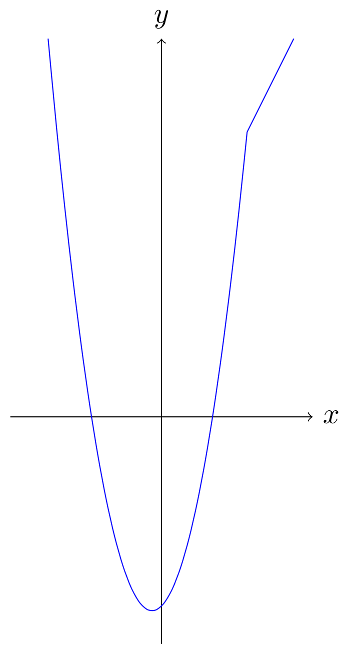


In light of the above discussion we ensure that both branches are equal at the point $x = t$. We therefore solve $2t + 3 = 2t^2 + t - 5$ or rather $0 = 2t^2 - t - 8$. Employing the quadratic formula we see

$$t = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(8)}}{2(2)}$$
$$t = \frac{1 \pm \sqrt{65}}{4}.$$

And so we have achieved both desired values for t. We can observe the following two graphs to see the continuity of f:



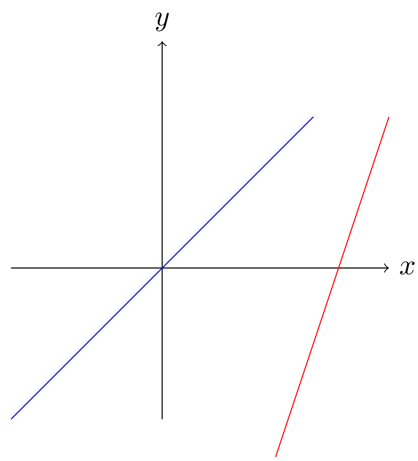


4. The absolute value function is defined as

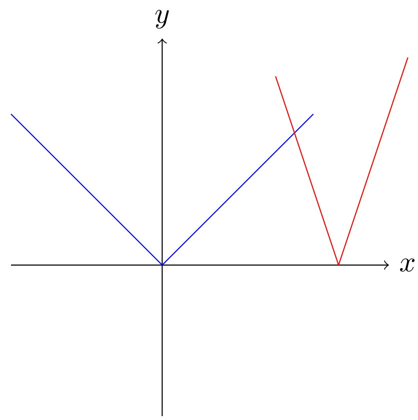
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}.$$

   (a) Plot $|x|$.
   (b) Plot $g(x) = |3x - 7|$.
   (c) Write g as a piecewise function.

   Solution: Bellow are the graphs of $y = x$ (blue) and $y = 3x - 7$ (red)



   The absolute value will make any negative value positive and so we see the graphs of $y = |x|$ (blue) and $y = |3x - 7|$ (red)



To find $g(x)$ as a piecewise function we note that if $3x - 7 \geq 0$ than the absolute value will have no effect. That is $g(x) = 3x - 7$ when $3x - 7 \geq 0$ or rather when $x \geq 7/3$. On the other hand if $3x - 7$ is negative than $|3x - 7| = -(3x - 7)$ a positive value. Therefore $g(x) = -(3x - 7)$ when $x < 7/3$. Putting this together reads

$$g(x) = \begin{cases} 3x - 7, & x \geq 7/3 \\ -(3x - 7), & x < 7/3 \end{cases}.$$

Alternatively one can pull the branches off the graph by finding the equation of each line and where they hit the x-axis.

5. Consider the function

$$f(x) = \frac{1}{\sqrt{x+1} - 1}.$$

(a) Find the *domain* of function f.

(b) The *range* of a function is the set of all possible output values of said function. Determine the range of f.

Solution: In order for $f(x)$ to make sense we require that $x + 1 \geq 0$ and that $\sqrt{x+1} - 1 \neq 0$. The former avoids taking the square root of a negative number and the latter avoids dividing by zero. Our constraint that $x + 1 \geq 0$ implies that $x \geq -1$. For the other condition we solve

$$\begin{aligned} \sqrt{x+1} - 1 &\neq 0 \\ \sqrt{x+1} &\neq 1 \\ x + 1 &\neq 1^2 \\ x &\neq 0. \end{aligned}$$

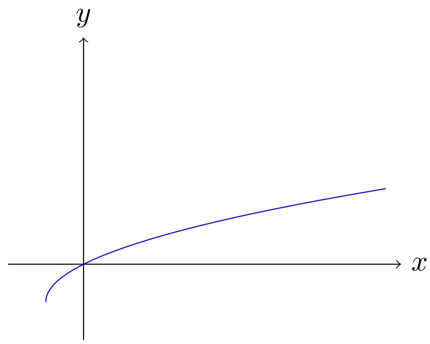
Putting our two conditions together we see the domain of f:

$$\{x \in \mathbb{R} : x \geq -1 \text{ and } x \neq 0\}$$

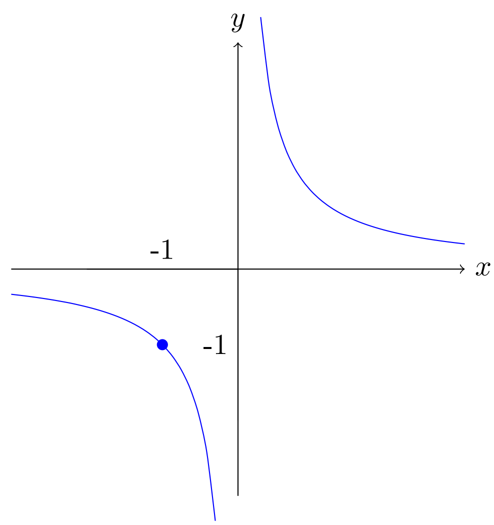
which can also be written as

$$[-1, 0) \cup (0, \infty).$$

To establish the range of f we must think about the possible output values for various parts of the function. Notice that f is a composition of the two functions $\sqrt{x+1}-1$ (inside) and $1/x$ (outside). It is our intention to find the range of $\sqrt{x+1}-1$ and then think about what happens when those output values are inputted to $1/x$. We know that $\sqrt{x+1} \geq 0$ since the output from a square root function is always positive. In fact $\sqrt{x+1}$ will take all positive values: the range of $\sqrt{x+1}$ is $[0, \infty)$. We can subtract 1 from both sides of our inequality to find the following information about the denominator: $\sqrt{x+1}-1 \geq -1$. Moreover, the range of the function $\sqrt{x+1}-1$ is $[-1, \infty)$. This fact can also be obtained from plotting the graph of $\sqrt{x+1}-1$ using transformations.



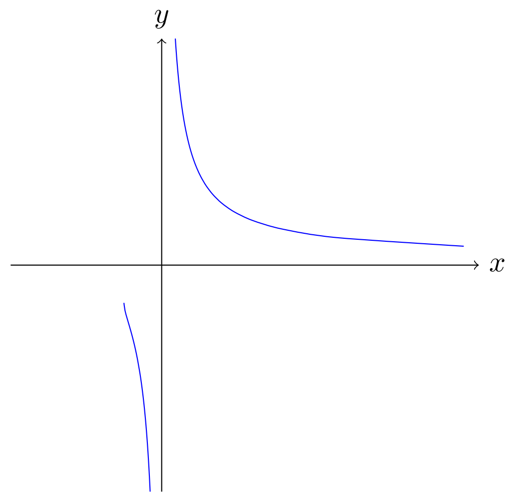
We now must think about what happens when we take all values in $[-1, \infty)$ and feed them into the function $1/x$. Observing the graph of $1/x$ we see that input values of $(0, \infty)$ will yield as output all values in $(0, \infty)$.



We also notice that for input values of $[-1, 0)$ the function $1/x$ will output all values in $(-\infty, -1]$. Putting everything together we see the range of f as

$$(-\infty, -1] \cup (0, \infty).$$

By the way, the graph of $f(x)$ looks like this:


y
x