Math 190 Homework 2: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:

1. Using the relevant graphs/triangles/unit circle explain why

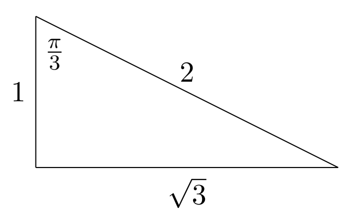
$$\cos\left(\frac{8\pi}{3}\right) = -\frac{1}{2}.$$

Solution: We first notice that $8\pi/3$ is larger than 2π. In fact $8\pi/3 = 2\pi + 2\pi/3$. Since the period of cosine is 2π we see that

$$\cos\left(\frac{8\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right).$$

Note that adding an additional 2π is equivalent to spinning around the unit circle exactly once. Now, in light of the special triangle we know that

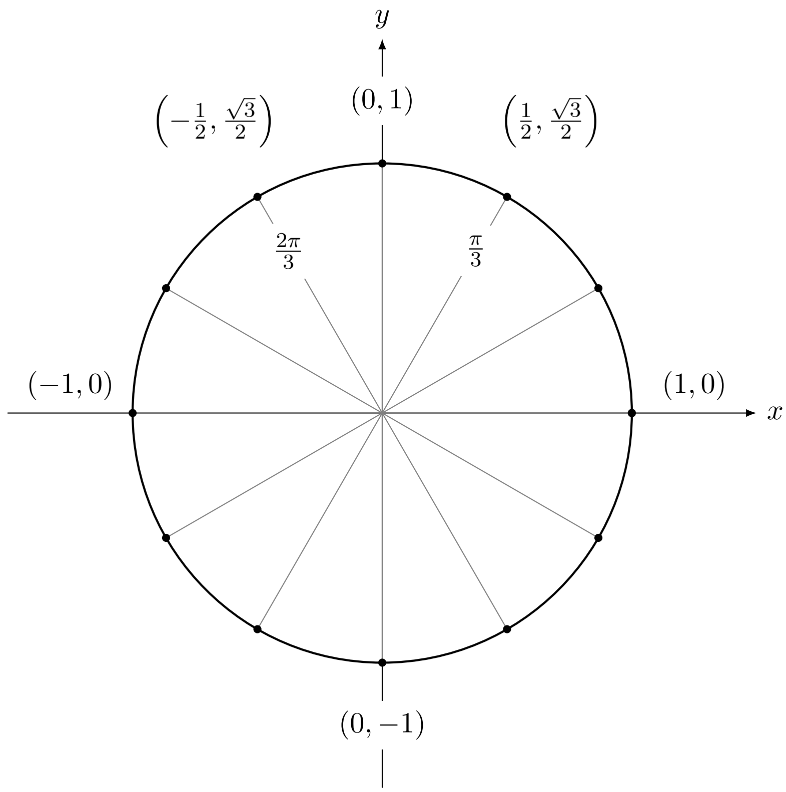
$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$



Observing the unit circle achieve that

$$\cos\left(\frac{8\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

the desired result.



2. Find all $x \in [0, 2\pi)$ satisfying

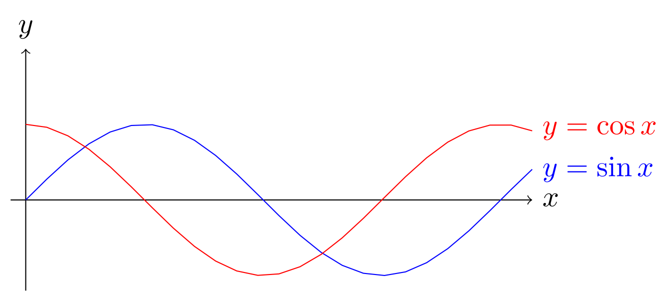
$$\cos x - \sin x = 0.$$

Ensure your answer is fully justified. Consider supporting your answer with a picture.

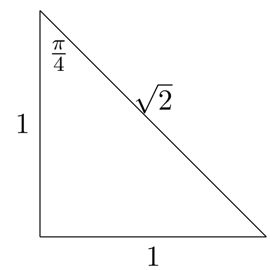
Solution: We can rewrite our equation as

$$\cos x = \sin x.$$

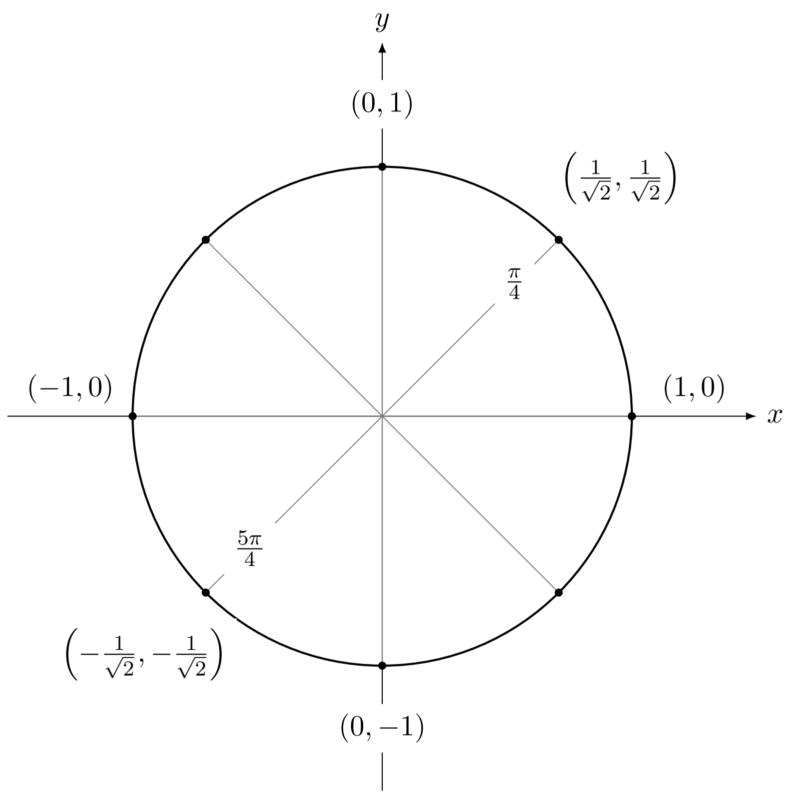
In this way we seek values where cosine and sine are equal. If we plot the graphs on top of each other we see that we expect exactly two solutions; one in the first quadrant and one in the third.



Cosine represents the x value of a point on the unit circle while sine represents the y value. It is maybe then not surprising that an angle of $\pi/4$ yields equal values for sine and cosine. Observe also the relevant special triangle.



So we have $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$ and so one of our intersection points is $x = \pi/4$. We now want the intersection point in the third quadrant. An inspection of the unit circle reveals that the desired point is $x = 5\pi/4$. Therefore our two solutions are $x = \pi/4$ and $x = 5\pi/4$.



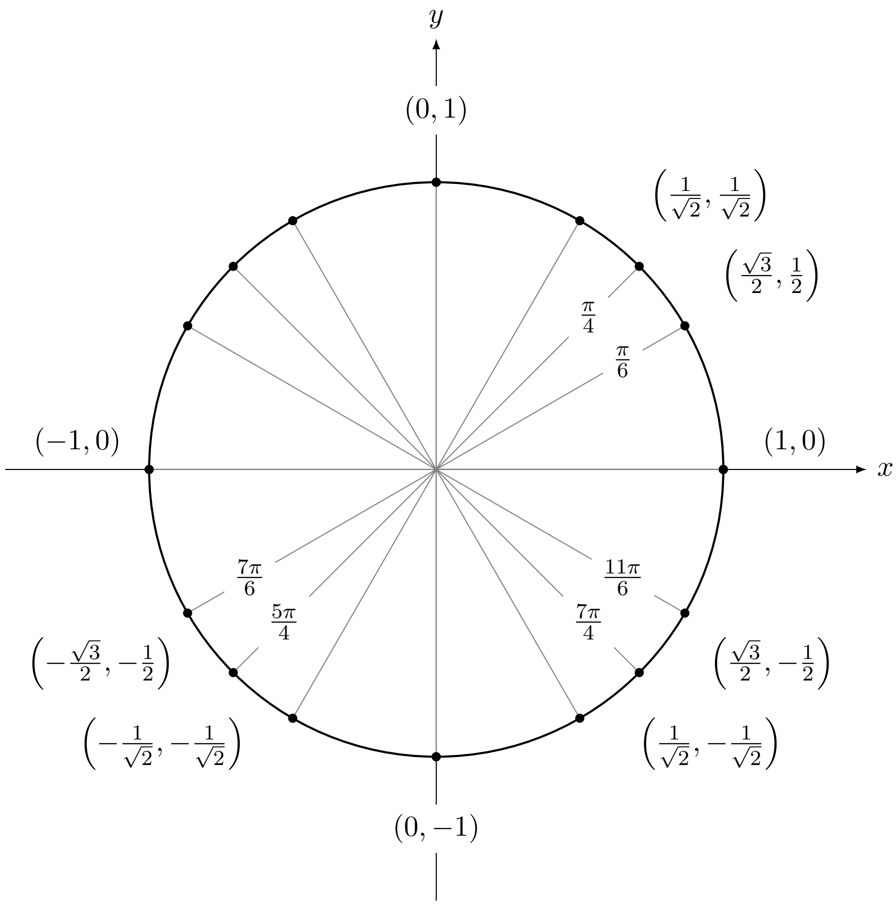
3. Find all $x \in [0, 2\pi)$ satisfying

$$2\sqrt{2}\sin^2 x + (\sqrt{2}+2)\sin x + 1 = 0.$$

Solution: We factor using decomposition. Observe

$$\begin{aligned} 0 =& 2\sqrt{2}\sin^2 x + \sqrt{2}\sin x + 2\sin x + 1 \\ =& \sqrt{2}\sin x(2\sin x + 1) + (2\sin x + 1) \\ =& (\sqrt{2}\sin x + 1)(2\sin x + 1) \end{aligned}$$

and so we wish to solve both $\sin x = -1/\sqrt{2}$ and $\sin x = -1/2$. For the first we see $x = 5\pi/4$ and $x = 7\pi/4$. For the second we see $7\pi/6$ and $11\pi/6$. Again, we achieve these values via special triangles and/or the unit circle.



4. Consider the following functions

$$g(x) = 2\sin(2016x)$$

and

$$f(x) = \begin{cases} x^3 - 7x & \text{if} \quad x > 2 \\ 7 & \text{if} \quad -2 \leq x \leq 2 \\ e^{4x} & \text{if} \quad x < -2 \end{cases}.$$

Determine the *range* of the function $f(g(x))$. Ensure your answer is fully justified.
Solution: We know the range of $\sin x$ is $[-1, 1]$. That is

$$-1 \leq \sin x \leq 1.$$

The constant inside the sine only changes the period, not the amplitude so $-1 \leq \sin(2016x) \leq 1$ holds and therefore

$$-2 \leq 2\sin(2016x) \leq 2.$$

With the range of $g(x)$ in hand we imagine feeding the output of g into $f(x)$. Since $-2 \leq g(x) \leq 2$ no matter the value of x we will always select the middle branch of f. In this way

$$f(g(x)) = 7$$

and so the range of $f(g(x))$ is $\{7\}$.

5. Find all (real) zeros of the function

$$h(x) = \cos\left(\frac{1}{x}\right).$$

Solution: There are many real zeros of this complicated function. To simplify things at first we make the substitution $u = 1/x$. In this way we find the zeros of $\cos u = 0$. We know these zeros are

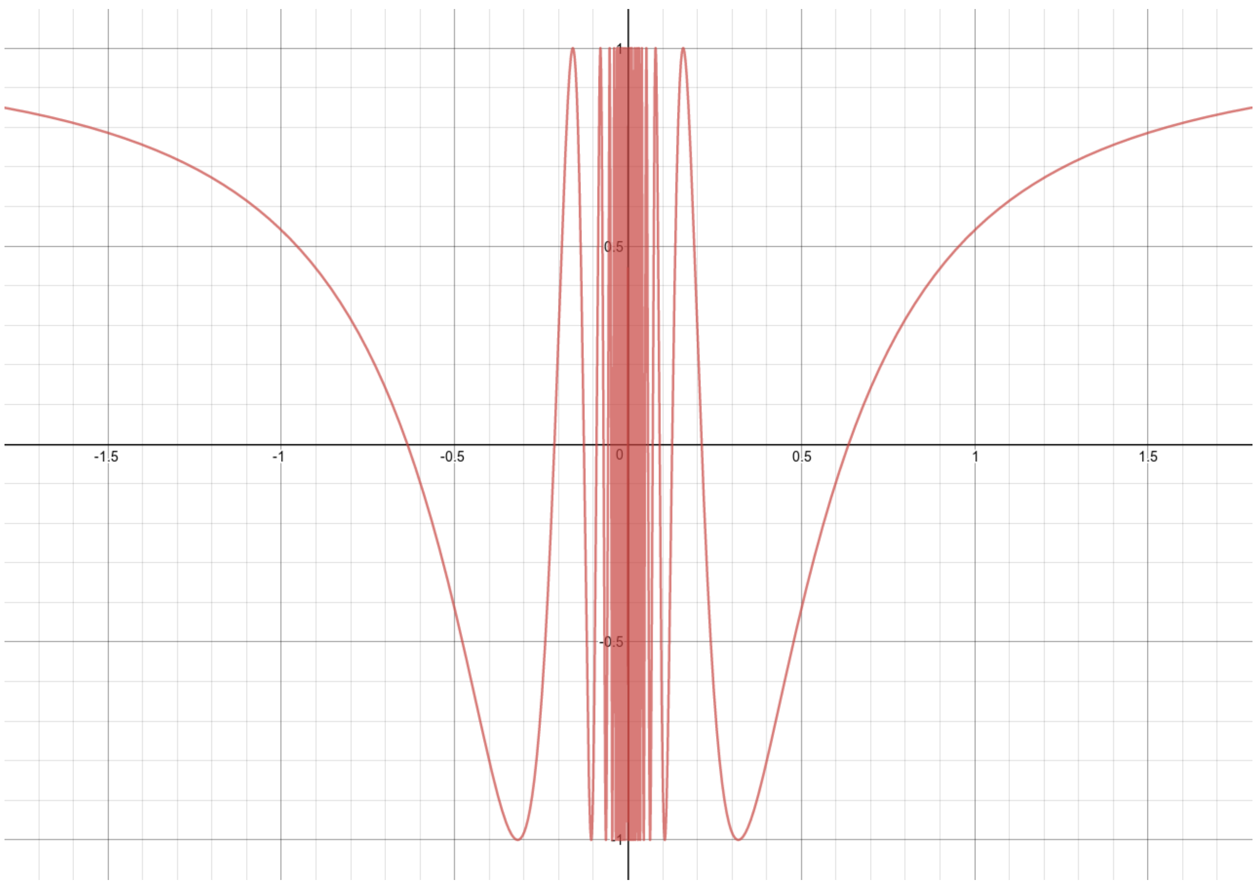
$$u = \frac{\pi}{2} + n\pi$$

where n is an integer. In this way our desired x values are

$$x = \frac{1}{\pi/2 + n\pi}$$

where n is an integer.

If we investigate these values a little we see that as n gets bigger the zero, x, gets smaller. In this way we expect to have many very small zeros. In fact we will have infinitely many zeros in any interval containing the origin. We can get a plot of the graph (from Desmos) to observe this behaviour (next page).



-1.5
-1
-0.5
0
0.5
1
1.5
0.5
-0.5
-1