Math 190 Homework 3: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:

1. Find all x satisfying

$$\ln(-x+1) + \ln(6) = e$$

Solution: Let us use logarithm rules to rewrite this equation in the following form

$$\begin{aligned} \ln(-x+1) + \ln(6) &= e \\ \ln\left(6(-x+1)\right) &= e \\ \ln(-6x+6) &= e. \end{aligned}$$

At this point we consider both sides to the power of e and so cancel the logarithm, recalling that e^x and $\ln x$ are inverses. Observe

$$\begin{aligned} e^{\ln(-6x+6)} &= e^e \\ -6x+6 &= e^e \\ -6x &= e^e - 6 \\ x &= \frac{6-e^e}{6} \end{aligned}$$

where we have finished the problem after some simple manipulation.

2. Consider

$$f(x) = e^{x\ln(x+2)}.$$

(a) Find the domain of f.

(b) Find all real x so that $f(x) = 1$.

Solution: (a) To find the domain of this function let us investigate the functions components. There is no problem with the function e^x; it is defined for all values of x. Additionally, multiplying a number by x will produce a sensible number. In this way we direct our attention to $\ln(x+2)$ in terms of domain considerations. The function $\ln(x+2)$ does not exist (in the sense of real numbers) for all input values of x. Recall that the input of the logarithm must be positive. Therefore we insist that $x+2>0$ or rather that $x>-2$ and so behold the range

$$\{x \in \mathbb{R} : x > -2\}$$

which can also be denoted as

$$(-2, \infty).$$

(b) We wish to find real x satisfying

$$1 = f(x) = e^{x \ln(x+2)}.$$

Let us start by taking the natural logarithm of both sides. This yields

$$\ln(1) = \ln\left(e^{x \ln(x+2)}\right).$$

Recalling that $\ln(1) = 0$ and that $\ln x$ and e^x are function inverses we find

$$0 = x \ln(x+2).$$

The above equation will be zero under two circumstances. Firstly, if $x = 0$ and secondly if $\ln(x+2) = 0$. The second equation is satisfied when (and only when) $x + 2 = 1$ since 1 is the only zero of $\ln x$ (think of the graph of $\ln x$). In this way we achieve our two solutions $x = 0$ and $x = -1$. Note that both of our solutions are in the domain of $f(x)$.

3. Find all real x satisfying

$$e^{2x} + e^x - 6 = 0.$$

Solution: The key to this problem is recognizing this equation as a quadratic equation. Using our exponent rules we can rewrite the above as

$$(e^x)^2 + e^x - 6 = 0.$$

If we like we can make the substitution $u = e^x$ to make the idea especially clear

$$\begin{aligned} u^2 + u - 6 &= 0 \\ (u+3)(u-2) &= 0 \\ (e^x - 2)(e^x + 3) &= 0 \end{aligned}$$

where we have achieved the above after some factoring and eliminating u. We therefore investigate both $e^x - 2 = 0$ and $e^x + 3 = 0$. Let us solve the first, we rearrange and then apply the natural logarithm

$$\begin{aligned} e^x - 2 &= 0 \\ e^x &= 2 \\ \ln(e^x) &= \ln(2) \\ x &= \ln(2). \end{aligned}$$

We have now achieved one solution, $x = \ln 2$. For the second equation, $e^3 + 3 = 0$, let us first rearrange to see

$$\begin{aligned} e^x + 3 &= 0 \\ e^x &= -3. \end{aligned}$$

Now, let's stare at this equation for a minute. If we think about the graph of e^x we notice that the range of e^x is $(0, \infty)$ or rather $e^x > 0$. In this way there is no (real) solution to $e^x = -3$. Alternatively, one may notice that if you try to take the natural logarithm of both side we would find $x = \ln(-3)$ which does not exist (as a real number). Based on our above discussion we have only the one solution: $x = 2$.

4. Many natural phenomena obey power rules. That is

$$Y = CX^m$$

where C and m are positive constants which depend on the particular application. For example in physics we have the Stephan-Boltzmann equation where Y is the power emitted by a star with temperature X. In forestry we have models of tree size distribution where Y is the number of trees with stem size X. Other examples include frequency of words in most languages, population of cities, and rate of reaction in chemistry.

(a) Let $y = \ln Y$ and $x = \ln X$. Express y in terms of x assuming that $Y = CX^m$. Note that C and m are fixed constants.

(b) Suppose we made a plot of y as a function of x. What would the graph look like?

Solution: (a) Let us start with $Y = CX^m$. Since we want to introduce x and y we take the natural logarithm of both sides

$$\ln Y = \ln \left(CX^m \right).$$

The left side is exactly y. For the right side we apply log rules:

$$\begin{aligned} y &= \ln C + \ln X^m \\ &= \ln C + m \ln X. \end{aligned}$$

At this stage we have recognized $x = \ln X$ in our equation and so we make the substitution

$$y = \ln C + mx.$$

We have now achieved y as a function of x as desired.

(b) Since C is a fixed constant the number $\ln C$ is also a fixed constant. We can recognize our equation for y as taking the form $y = mx + b$ where m is m and b is $\ln C$. In this way we see that our equation is the equation of a line with y-intercept $\ln C$ and slope m.

5. In this problem you will prove the identity

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

as seen in class. First let $z_1 = \log_b(x)$ and $z_2 = \log_b(y)$. Rewrite these two equations using exponents instead of logarithms. Use your knowledge of exponent rules to manipulate the equations until you achieve $z_1 + z_2 = \log_b(xy)$. Make sure that you explain each step.

Solution: There are several ways to formulate the proof. Here is one way. Let $z_1 = \log_b(x)$ and $z_2 = \log_b(y)$. That is to say that $b^{z_1} = x$ and $b^{z_2} = y$. Let us now consider the product

$$xy = b^{z_1} b^{z_2}$$

and use exponent rules to see

$$xy = b^{z_1 + z_2}.$$

Switching this exponential equation back to a logarithm equation gives

$$\log_b(xy) = z_1 + z_2.$$

Recalling the definition of z_1 and z_2 we arrive at the desired identity

$$\log_b(xy) = \log_b(x) + \log_b(y).$$

6. **Bonus** Prove the other two logarithm identities.

Solution: First we show $\log_b(x/y) = \log_b(x) - \log_b(y)$. This proof proceeds in the same manner as the previous. Let $z_1 = \log_b(x)$ and $z_2 = \log_b(y)$ that is to say $b^{z_1} = x$ and $b^{z_2} = y$. Consider now the quotient and use the exponent rule to see

$$\frac{x}{y} = \frac{b^{z_1}}{b^{z_2}} = b^{z_1 - z_2}.$$

Switching back to a logarithm equation we have

$$\log_b\left(\frac{x}{y}\right) = z_1 - z_2 = \log_b(x) - \log_b(y).$$

With the required result in hand the proof is complete.

Now the third identity: $\log_b(x^p) = p\log_b x$. To start let $z = p\log_b(x)$. Let us manipulate this equation using our knowledge of logarithms and exponent laws:

$$\begin{aligned} \frac{z}{p} &= \log_b(x) \\ b^{z/p} &= x \\ (b^z)^{1/p} &= x \\ b^z &= x^p \\ z &= \log_b(x^p). \end{aligned}$$

We have therefore achieved

$$p\log_b(x) = \log_b(x^p)$$

as required. Note that our above manipulation requires that $p \neq 0$ (to avoid dividing by zero). However, the identity is obviously true when $p = 0$. Observe

$$\log_b(x^0) = \log_b(1) = 0$$

and

$$0 \cdot \log_b(x) = 0.$$