Math 190 Homework 4: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:
When asked to compute a limit in the following problems: Find the value of the limit if it exists. If the limit does not exist but you can assign the value ∞ or $-\infty$ to the limit do so. Otherwise, explain why the limit does not exist.

1. Compute $\lim_{x\to -3} \dfrac{\sqrt{x^2+7}-\sqrt{x+19}}{x+3}$.
   Solution: We have difficulty computing this limit directly since a substitution will yield "0/0". Our strategy is to multiply by the conjugate. After some algebraic manipulation (factoring) we are able to make some cancellation and finally substitute. Behold

$$\begin{aligned}
\lim_{x\to -3} \frac{\sqrt{x^2+7}-\sqrt{x+19}}{x+3} &= \lim_{x\to -3} \frac{\sqrt{x^2+7}-\sqrt{x+19}}{x+3} \cdot \frac{\sqrt{x^2+7}+\sqrt{x+19}}{\sqrt{x^2+7}+\sqrt{x+19}} \\
&= \lim_{x\to -3} \frac{x^2+7-(x+19)}{(x+3)(\sqrt{x^2+7}+\sqrt{x+19})} \\
&= \lim_{x\to -3} \frac{x^2-x-12}{(x+3)(\sqrt{x^2+7}+\sqrt{x+19})} \\
&= \lim_{x\to -3} \frac{(x+3)(x-4)}{(x+3)(\sqrt{x^2+7}+\sqrt{x+19})} \\
&= \lim_{x\to -3} \frac{(x-4)}{\sqrt{x^2+7}+\sqrt{x+19}} \\
&= \frac{(-3-4)}{\sqrt{(-3)^2+7}+\sqrt{-3+19}} \\
&= \frac{-7}{\sqrt{16}+\sqrt{16}} \\
&= -\frac{7}{8}.
\end{aligned}$$

   And so the limit exists and is given by the above value.

2. Compute $\lim_{x\to 2} \dfrac{x^2-x-2}{|x-2|}$.
   Solution: First we write $|x-2|$ as a piecewise function. That is

$$|x-2| = \begin{cases} x-2, & x \geq 2 \\ -(x-2), & x < 2 \end{cases}.$$

Since the formula for our function is different on either side of $x = 2$ we decide to compute the one sided limits. First the limit as we approach $x = 2$ from above 2 using the first branch of $|x - 2|$

$$\begin{aligned}\lim_{x\to 2^+} \frac{x^2 - x - 2}{|x-2|} &= \lim_{x\to 2^+} \frac{(x-2)(x+1)}{(x-2)} \\ &= \lim_{x\to 2^+} (x+1) \\ &= 2+1 \\ &= 3\end{aligned}$$

noting that we can cancel the $(x-2)$'s since $x \neq 2$ (x is just close to 2). For the other one sided limit (as we approach $x = 2$ from below 2)

$$\begin{aligned}\lim_{x\to 2^-} \frac{x^2 - x - 2}{|x-2|} &= \lim_{x\to 2^-} \frac{(x-2)(x+1)}{-(x-2)} \\ &= \lim_{x\to 2^-} -(x+1) \\ &= -(2+1) \\ &= -3\end{aligned}$$

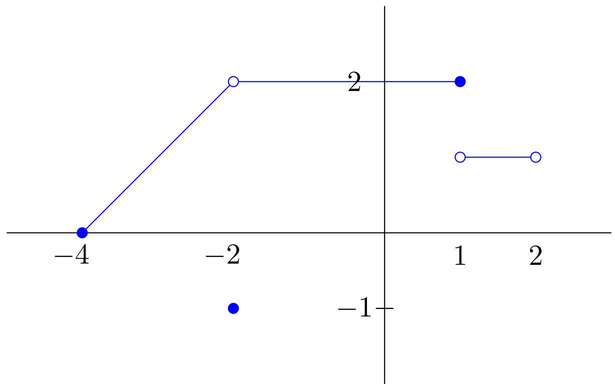
again, noting that $x \neq 2$. Observe now that

$$\lim_{x\to 2^+} \frac{x^2 - x - 2}{|x-2|} \neq \lim_{x\to 2^-} \frac{x^2 - x - 2}{|x-2|}.$$

In this way, since the one sided limits are not equal, we conclude that the full limit does not exist.

3. Draw the graph of a function $f(x)$ satisfying the following properties (you do not have to come up with an equation for your graph).
   - The domain is $\{x \in \mathbb{R} : -4 \leq x < 2\}$.
   - $f(-2) = -1$
   - $\lim_{x\to -2} f(x) = 2$
   - $\lim_{x\to 1^-} f(x) = 2$
   - $\lim_{x\to 1^+} f(x) = 1$

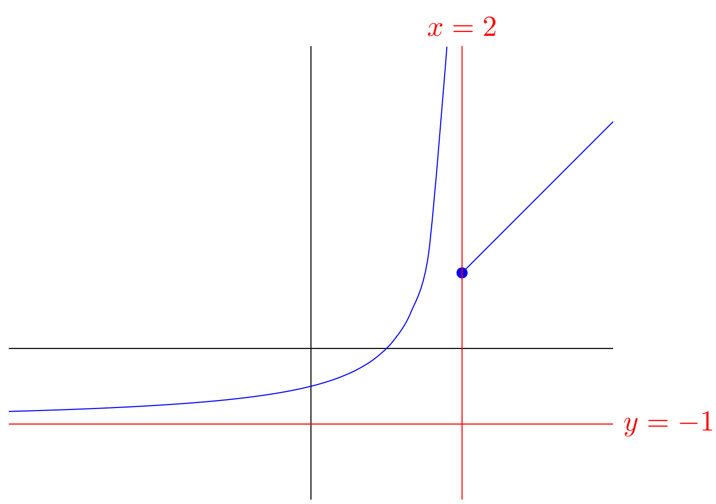
   Solution: We note that there are many functions that satisfy the above conditions (infinitely many in fact). We supply one such function below.



Note that our function is defined at all points between 4 and 2, including 4 but not including 2.

4. Draw the graph of a function $g(x)$ satisfying the following properties (you do not have to come up with an equation for your graph).
   - The domain is $\mathbb{R}$.
   - g has a horizontal asymptote at $y = -1$
   - $\lim_{x \to \infty} g(x) = \infty$
   - $\lim_{x \to 2^-} g(x) = \infty$

   Solution: Again, there are many possible solutions. Here is an example of one such graph (blue). We have also included the graphs of $y = -1$ and $x = 2$, the asymptotes. Note that we have defined $f(2)$ which must take a value in order to have the domain be $\mathbb{R}$.



5. Consider the function

$$h(x) = e^{-x} \cos(3x).$$

   (a) Explain what happens to e^{-x} as x approaches ∞.
   (b) Explain what happens to $\cos(3x)$ as x approaches ∞.

(c) Using your answers from (a) and (b) explain what happens to the values of $h(x)$ as $x \to \infty$. In this way you can suggest a value for

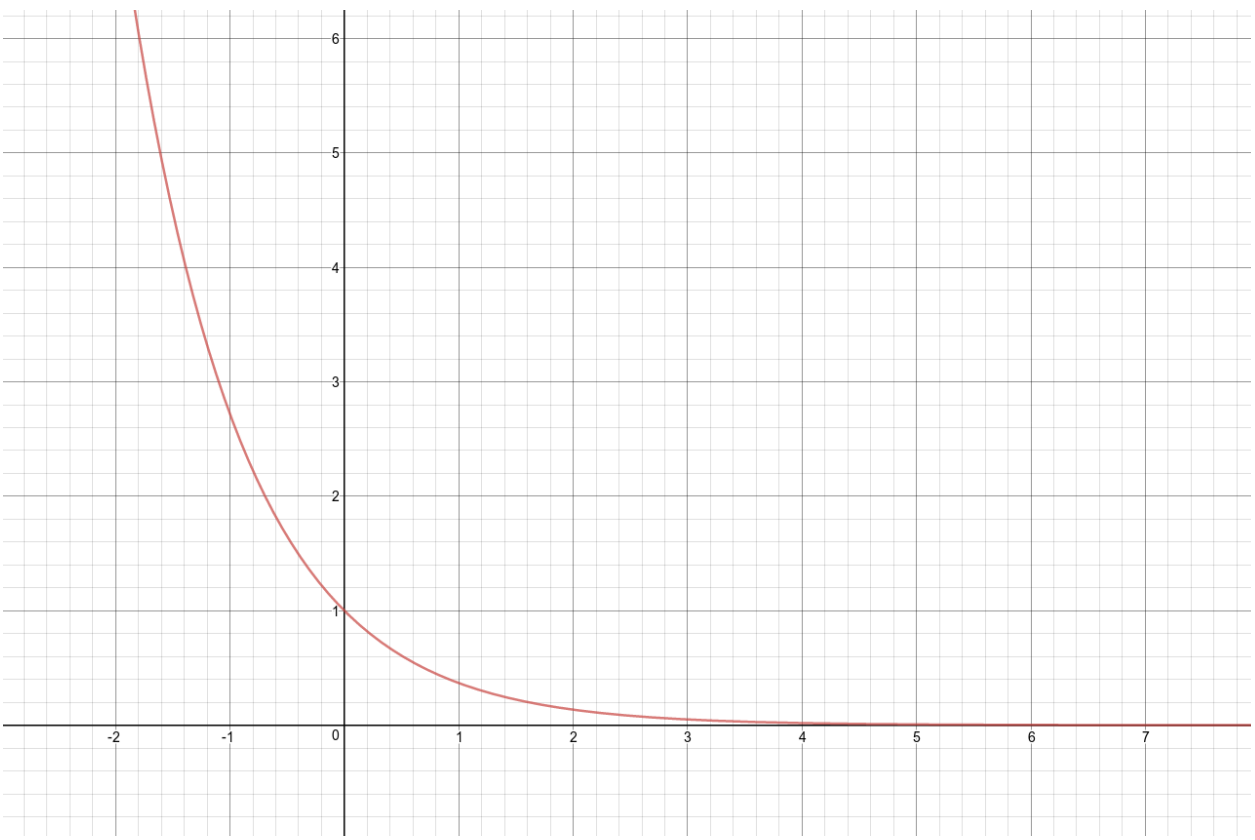
$$\lim_{x\to\infty} e^{-x}\cos(3x).$$

(d) Find the zeros of $h(x)$. Make a rough sketch of the graph indicating the zeros as well as any asymptotes.

Solution: (a) For large values of x the value $-x$ will be a large negative number and so in turn e^{-x} will be a small positive number. In this way $e^{-x} \to 0$ as $x \to \infty$. That is as x approaches ∞, e^{-x} approaches 0. We could also write

$$\lim_{x\to\infty} e^{-x} = 0.$$

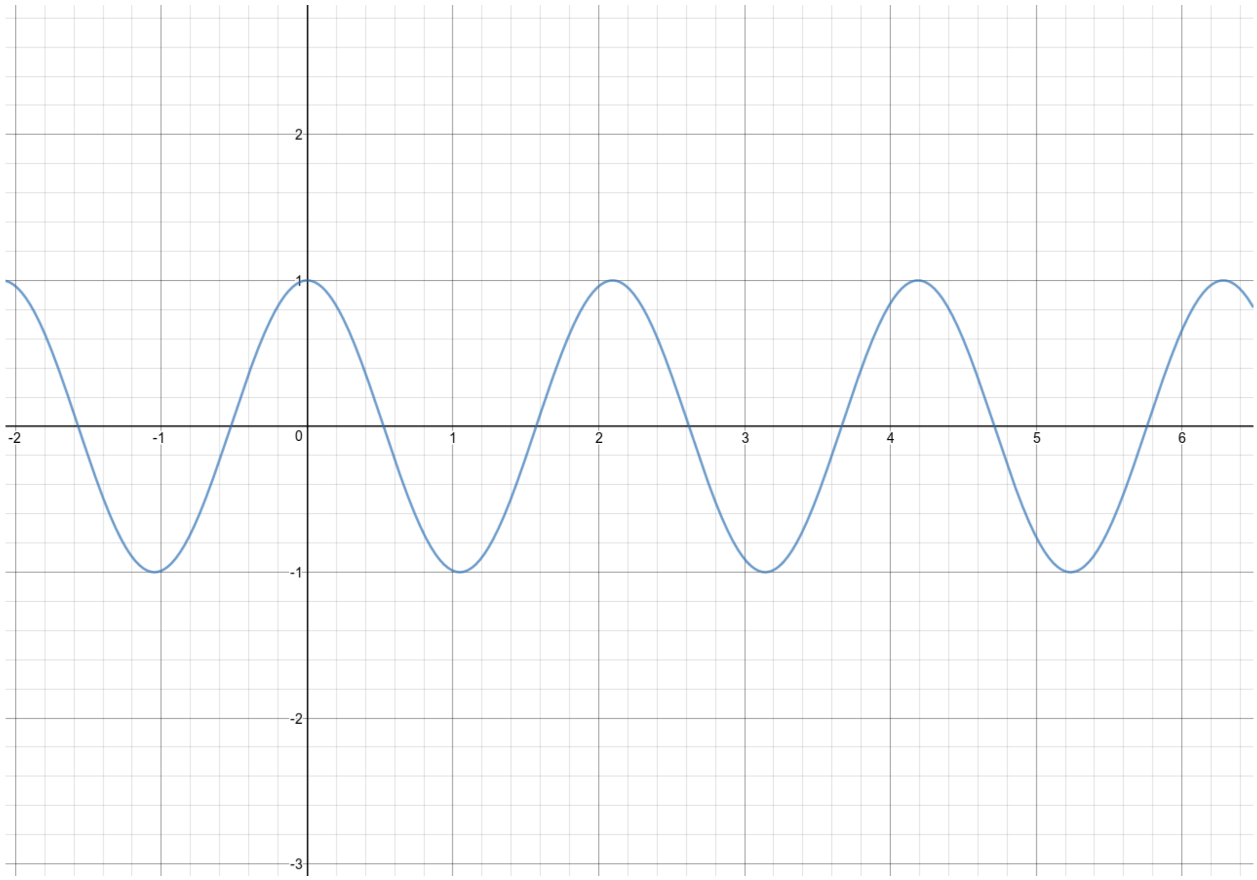
This fact can also be seen from the graph of e^{-x}



(b) As x increase the values for $\cos(3x)$ continue to oscillate between -1 and 1. There is no one number that the function approaches and so $\lim_{x\to\infty}\cos(3x)$ does not exist. We can also observe the graph of $y = \cos(3x)$.

(c) We now consider the limit

$$\lim_{x\to\infty} e^{-x}\cos(3x).$$



As discussed the first factor e^{-x} is approaching zero while the second is continually oscillating and bounded between -1 and 1. If we imagine mutiplying increasingly smaller numbers by a number that is staying between -1 and 1, overall our number will become very small. In this way we know that

$$\lim_{x\to\infty} e^{-x}\cos(3x) = 0$$

and so $h(x)$ has a horizontal asympotote at $y = 0$.

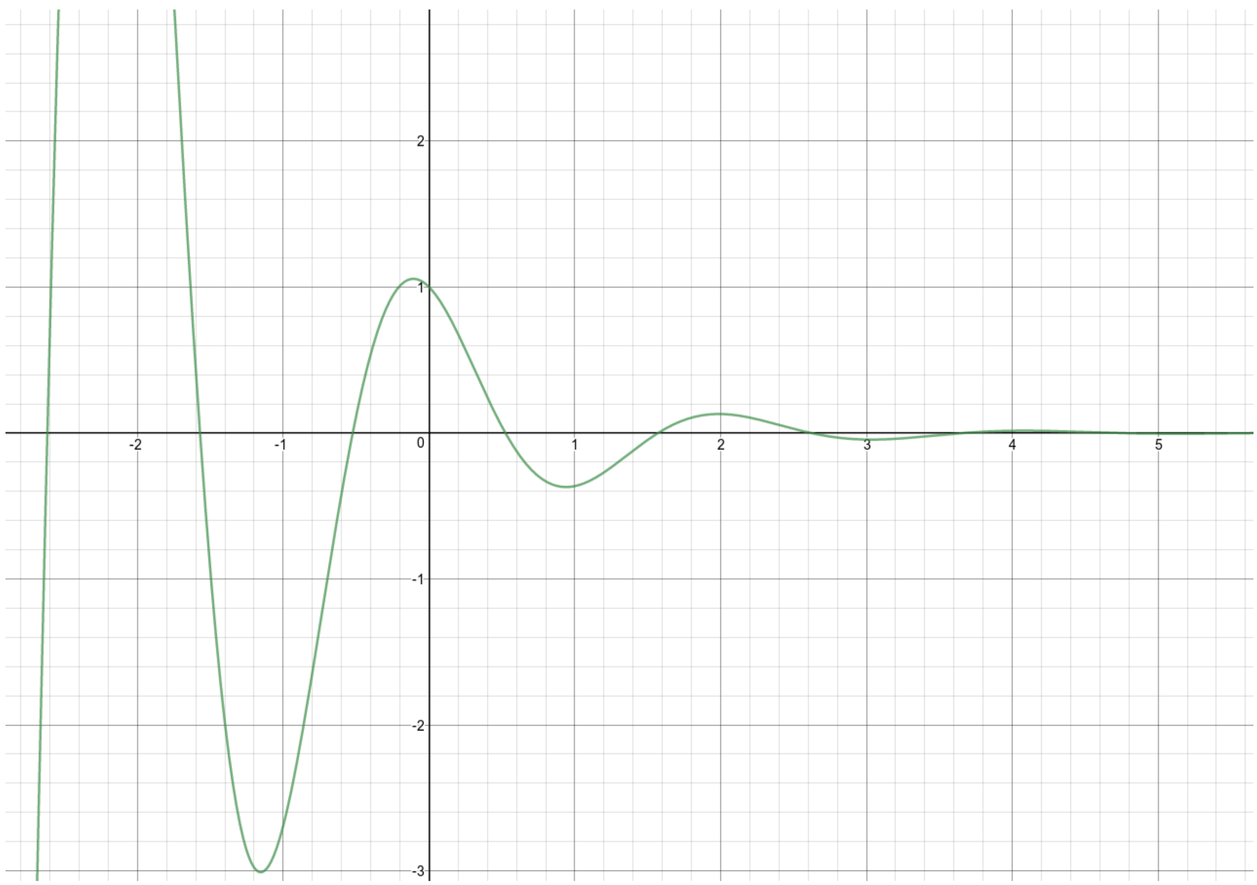
(d) This graph will have infinitely many zeros. If we set $0 = e^{-x}\cos(3x)$ we investigate both $0 = e^{-x}$ and $0 = \cos(3x)$. We however note that e^{-x} is always positive and so $e^{-x} \neq 0$. In this way we can only get zeros from $0 = \cos(3x)$. To solve this equation we can make the substitution $u = 3x$ to see $0 = \cos(u)$. We know the solutions for u will be

$$u = \frac{\pi}{2} + n\pi$$

for any integer n. Noting then that $x = u/3$ we achieve

$$x = \frac{\pi}{6} + n\frac{\pi}{3}$$

for any integer n. Therefore our function $h(x)$ will cross its asymptote infinitely many times. No matter how large x gets there will still be infinitely many more crossings to come. We can think of e^{-x} as controlling the amplitude of the cosine function. As x gets larger the amplitute is getting smaller so our crests are becoming smaller and smaller.



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-2
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-2
-1
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5