Math 190 Homework 5: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page. Enjoy this short assignment for this short week!

Questions:

1. Final all vertical and horizontal asymptotes of the following functions. Ensure you compute (and show the computation of) all the relevant limits.

   (a) $\dfrac{x-4}{x^2-2x-8}$

   (b) $\dfrac{e^x}{e^x+2}$

   Solution: (a) Let us first look for vertical asymptotes. We rewrite the function as follows

$$\frac{x-4}{x^2-2x-8} = \frac{x-4}{(x-4)(x+2)}.$$

Observing the denominator we identify the following candidates for vertical asymptotes: $x = 4$ and $x = -2$. We first consider be behaviour around $x = 4$ by computing the one sided limits

$$\lim_{x\to 4^+} \frac{x-4}{(x-4)(x+2)} = \lim_{x\to 4^+} \frac{1}{x+2} = \lim_{x\to 4^+} \frac{1}{4+2} = \frac{1}{6}$$
$$\lim_{x\to 4^-} \frac{x-4}{(x-4)(x+2)} = \lim_{x\to 4^-} \frac{1}{x+2} = \lim_{x\to 4^-} \frac{1}{4+2} = \frac{1}{6}.$$

And so, since our function approaches a number as x approaches 4, we have no vertical asymptote at $x = 4$. Note that we could have just computed the full limit in this case. We now investigate $x = -2$. Observe the following one sided limits

$$\lim_{x\to -2^+} \frac{x-4}{(x-4)(x+2)} = \lim_{x\to -2^+} \frac{1}{x+2} = \infty, \quad \left(\frac{+}{+}\right)$$
$$\lim_{x\to -2^-} \frac{x-4}{(x-4)(x+2)} = -\infty, \quad \left(\frac{(-)}{(-)(-)}\right).$$

We have computed each limit differently to show two different methods. The limit can be established with or without cancelling. Since at least one of these limits is $+\infty$ or $-\infty$ we conclude that our function has a vertical asymptote at $x = -2$.

For horizontal asymptotes we consider the limits as $x \to \infty$ and $x \to -\infty$. Our trick is to divide each term by the highest power. There follows

$$\lim_{x\to\infty} \frac{x-4}{x^2-2x-8} = \lim_{x\to\infty} \frac{x/x^2-4/x^2}{x^2/x^2-2x/x^2-8/x^2} = \lim_{x\to\infty} \frac{1/x-4/x^2}{1-2/x-8/x^2} = \frac{0}{1} = 0$$

and similarly

$$\lim_{x\to-\infty} \frac{x-4}{x^2-2x-8} = \lim_{x\to-\infty} \frac{x/x^2-4/x^2}{x^2/x^2-2x/x^2-8/x^2} = \lim_{x\to-\infty} \frac{1/x-4/x^2}{1-2/x-8/x^2} = \frac{0}{1} = 0.$$

We now conclude that our function has a horizontal asymptote at $y = 0$.

(b) For vertical asymptotes we inspect the denominator. Note that e^x is always positive. In this was $e^x + 2 > 2$ for all values of x. Since the denominator is never close to zero there is no way to make the function blow up. (Note also that the numerator is well behaved and gives no possibility of a vertical asymptote). In this way we conclude that our function has no vertical asymptotes.

On now to horizontal asymptotes. First consider the limit

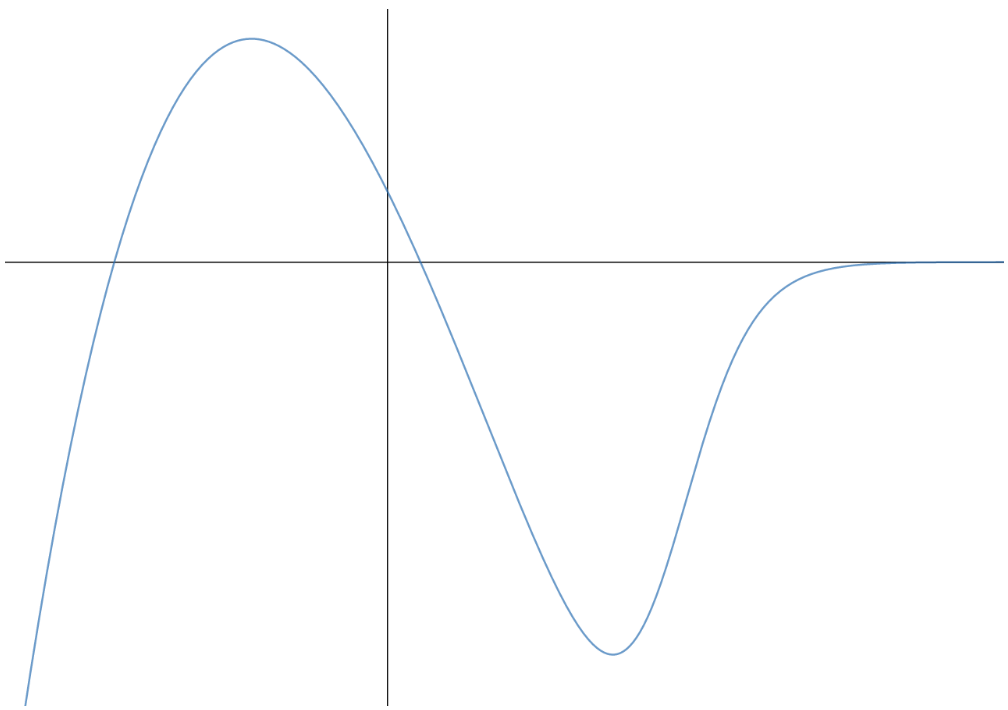
$$\lim_{x \to -\infty} \frac{e^x}{e^x + 2} = \frac{0}{0 + 2} = 0$$

noting that $e^x \to 0$ as $x \to -\infty$ (we could see this from the graph for example). In this way we have a horizontal asymptote at $y = 0$. For the other limit, we divide each term by e^x. In this way

$$\lim_{x \to \infty} \frac{e^x}{e^x + 2} = \lim_{x \to \infty} \frac{e^x/e^x}{e^x/e^x + 2/e^x} = \lim_{x \to \infty} \frac{1}{1 + 2/e^x} = \frac{1}{1 + 0} = 1.$$

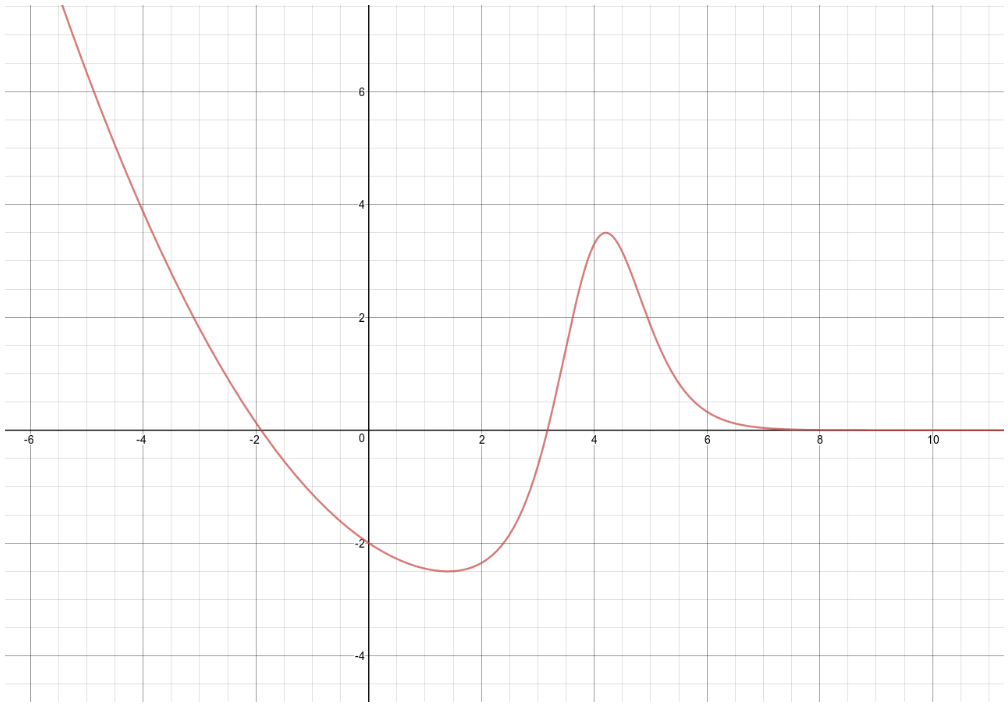
And so we see two horizontal asymptotes. One at $y = 0$ and the other at $y = 1$.

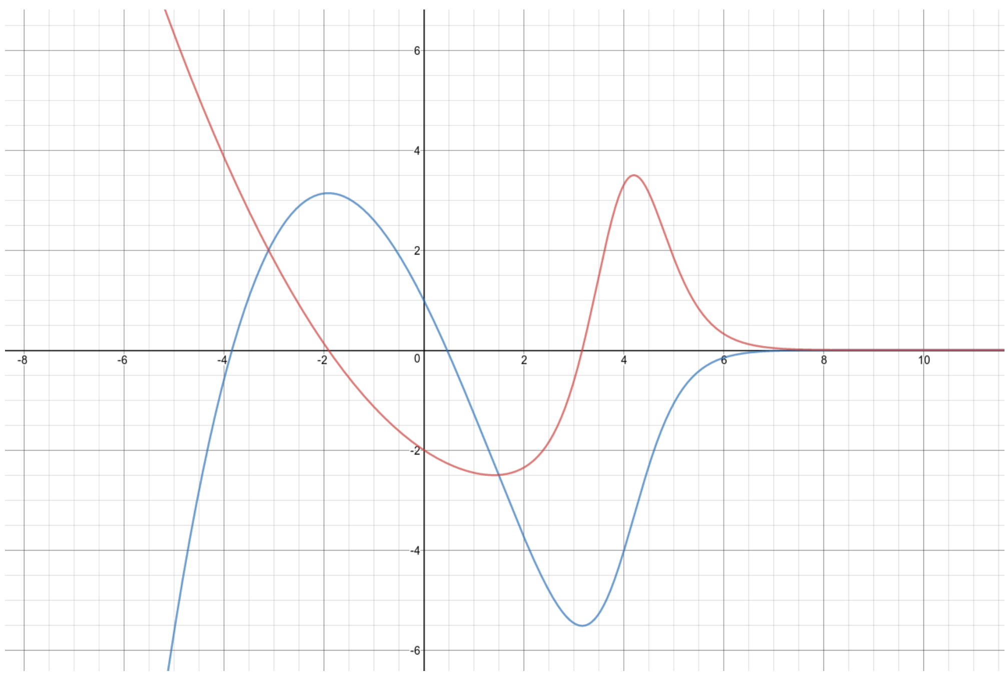
2. Consider the following graph of $f(x)$:



Sketch the graph of its derivative. Do so by considering the slope of the above graph and how it changes.

Solution: Here is a sketch of the derivative followed by a graph containing both the function and the derivative (not required).





In case you were wondering the equation of the original function is

$$\left(-\frac{1}{3}(x+3)^2+4+\frac{1}{15}x^3\right)\frac{(-\tanh(x-4)+1)}{2}$$

where $\tanh x$ (pronounced 'tanch') is the hyperbolic tangent function. The equation of its derivative is

$$-\frac{1}{2}\left(4+\frac{x^3}{15}-\frac{1}{3(3+x)^2}\right)\text{sech}^2(4-x)+\frac{1}{2}\left(\frac{x^2}{5}-\frac{1}{3}(2(3+x))\right)(1+\tanh(4-x))$$

where $\text{sech} x$ is the hyperbolic secant function.

3. Using the limit definition of the derivative (and not any other method) find the derivative of the function

$$g(x)=\frac{x-1}{x}$$

and use it to compute the equation of the tangent line to $g(x)$ at $x=3$.
Solution: Recall the definition of the derivative

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}.$$

We now compute with the intension of adding the fractions, cancelling the troublesome h in the bottom and finally substituting

$$\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{1}{h}\left(\frac{x+h-1}{x+h}-\frac{x-1}{x}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{(x+h-1)x}{(x+h)x}-\frac{(x+h)(x-1)}{(x+h)x}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{(x+h-1)x-(x+h)(x-1)}{(x+h)x}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{(x+h)x-x-(x+h)x+(x+h)}{x(x+h)}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{-x+x+h}{x(x+h)}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{h}{x(x+h)}\right)\\
&= \lim_{h\to 0}\left(\frac{1}{x(x+h)}\right)\\
&= \frac{1}{x(x+0)}\\
&= \frac{1}{x^2}.
\end{aligned}$$

With the derivative in hand we compute the equation of the tangent line at $x=3$. The slope of the tangent line will be given by $g'(3)=1/3^2=1/9$. We also have a point on the tangent line, that is $(3,g(3))=(3,2/3)$. Therefore the equation of our tangent line is

$$y-\frac{2}{3}=\frac{1}{9}(x-3)$$

in slope point form.