Math 190 Homework 6: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:

1. For the following problems find the derivative of the given function.
   (a) $5x^4 - \dfrac{2}{\sqrt[3]{x}} + x^e - e^x$
   (b) $x^{3/2}\sin x$
   (c) $\dfrac{\cos x}{e^x}$

   Solution: We compute the required derivatives

   (a) $\left(5x^4 - \dfrac{2}{\sqrt[3]{x}} + x^e - e^x\right)' = \left(5x^4 - 2x^{-1/3} + x^e - e^x\right)' = 20x^3 + \dfrac{2}{3}x^{-4/3} + ex^{e-1} - e^x$

   (b) $\left(x^{3/2}\sin x\right)' = \dfrac{3}{2}x^{1/2}\sin x + x^{3/2}\cos x$

   (c) $\left(\dfrac{\cos x}{e^x}\right)' = \dfrac{-\sin x e^x - \cos x e^x}{(e^x)^2} = \dfrac{-\sin x e^x - \cos x e^x}{e^{2x}} = \dfrac{-\sin x - \cos x}{e^x}$

   Note that simplification is not necessary.

2. Find the equation of the tangent line to

$$h(x) = \frac{x^2e^x}{e^x+1}$$

   at the point $x = \ln(2)$.

   Solution: To find the equation of the tangent line we will need the slope. To find the slope we compute the derivative. Observe

$$\begin{aligned} h'(x) &= \frac{(x^2e^x)'\cdot(e^x+1) - x^2e^x\cdot e^x}{(e^x+1)^2} \\ &= \frac{(2xe^x + x^2e^x)(e^x+1) - x^2e^xe^x}{(e^x+1)^2} \end{aligned}$$

   where we have used two steps to emphasize that we have used product rule inside of quotient rule. The slope of our tangent line at $x = \ln(2)$ will be given by

$$\begin{aligned} h'(\ln(2)) &= \frac{(2\ln(2)e^{\ln(2)} + (\ln(2))^2e^{\ln(2)})(e^{\ln(2)}+1) - (\ln(2))^2e^{\ln(2)}e^{\ln(2)}}{\left(e^{\ln(2)}+1\right)^2} \\ &= \frac{(4\ln 2 + 2(\ln 2)^2)(2+1) - 4(\ln 2)^2}{(2+1)^2} \\ &= \frac{12\ln 2 + 6(\ln 2)^2 - 4(\ln 2)^2}{9} \\ &= \frac{12\ln 2 + 2(\ln 2)^2}{9} \end{aligned}$$

noting that $e^{\ln 2} = 2$. We also need the y coordinate corresponding to $x = \ln 2$. That is

$$h(\ln 2) = \frac{2(\ln 2)^2}{3}.$$

And so, the equation of our tangent line, in slope-point form, is

$$y - \frac{2(\ln 2)^2}{3} = \left(\frac{12\ln 2 + 2(\ln 2)^2}{9}\right)(x - \ln 2).$$

3. Consider the function

$$f(x) = x^{1/3}.$$

   (a) Find the domain of $f(x)$.
   (b) Find the derivative of $f(x)$ using power rule. What is the domain of $f'(x)$?
   (c) Find all vertical asymptotes of the derivative $f'(x)$. Include the relevant one sided limits.
   (d) Sketch the graph of $f(x) = x^{1/3}$. Try to draw the tangent line at $x = 0$. What is the slope? Explain this behaviour in connection with your answer from part (c).

   Solution: (a) The domain of $f(x)$ is $\mathbb{R}$. This can also be written as $\{x \in \mathbb{R}\}$ or $(-\infty, \infty)$.

   (b) We compute the derivative

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

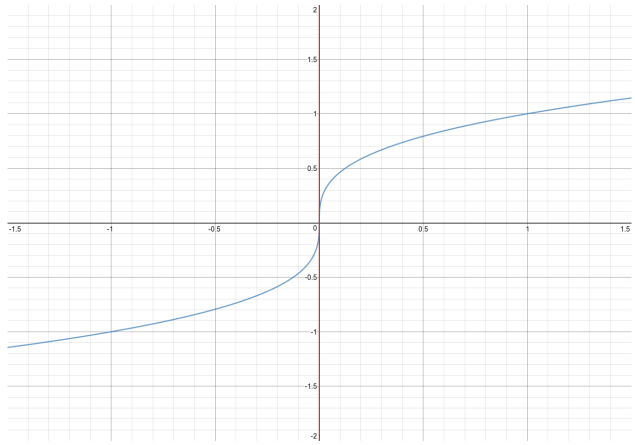
   and observe the domain of the derivative to be $\{x \in \mathbb{R} : x \neq 0\}$ which can also be written as $(-\infty, 0) \cup (0, \infty)$.

   (c) Our candidate for a vertical asymptote is $x = 0$. We compute both one sided limits as follows

$$\lim_{x\to 0^+} \frac{1}{3\sqrt[3]{x^2}} = \infty$$
$$\lim_{x\to 0^-} \frac{1}{3\sqrt[3]{x^2}} = \infty$$

   noting that a negative number squared becomes positive. Since at least one of these limits (in fact both) are infinity we conclude that $f'(x)$ has a vertical asymptote at $x = 0$.

   (d) Observe the graph of $f(x) = x^{1/3}$ (blue) We have also included the tangent line (red) at $x = 0$. Observe that the tangent line is a vertical line with infinite slope. While the derivative itself does not exist at $x = 0$ the derivative does have a vertical asympotote at $x = 0$. Based on our above limit calculations the derivative is approaching infinity on both sides of zero. In terms of the function $f(x)$ itself this means that the slope of the tangent line is approaching infinity as $x \to 0$. It is then not surprising to find the vertical tangent line at $x = 0$.



4. (a) Recall the derivative of $x^{3/2}\sin x$ from Question 1(b). Using your answer, find the derivative of

$$e^x x^{3/2}\sin x$$

using product rule once.

(b) To solve the above problem (a) you could have also used the following *triple product rule*:

$$\frac{d}{dx}(fgh) = \frac{df}{dx}gh + f\frac{dg}{dx}h + fg\frac{dh}{dx}.$$

Use the triple product rule to differentiate

$$(3x-1)(4x^2+2)(x^{-3}+\sqrt{x}).$$

Solution: (a) We employ Product Rule once and use our previous derivative calculation. Behold

$$\begin{aligned}\left(e^x x^{3/2}\sin x\right)' &= e^x x^{3/2}\sin x + e^x\left(x^{3/2}\sin x\right)' \\ &= e^x x^{3/2}\sin x + e^x\left(\frac{3}{2}x^{1/2}\sin x + x^{3/2}\cos x\right) \\ &= e^x x^{3/2}\sin x + e^x\frac{3}{2}x^{1/2}\sin x + e^x x^{3/2}\cos x.\end{aligned}$$

Note that we could have also used the tripple product rule to the same effect.

(b) Let us differentiate using the tripple produce rule. There follows

$$\left((3x-1)(4x^2+2)(x^{-3}+\sqrt{x})\right)' =$$

$$3(4x^2+2)(x^{-3}+\sqrt{x}) + (3x-1)(8x)(x^{-3}+\sqrt{x}) + (3x-1)(4x^2+2)\left(-3x^{-4}+\frac{1}{2}x^{-1/2}\right).$$

5. Not every function is *differentiable* (has a derivative) at every point. Consider the function $f(x) = |x|$. Let us try to investigate the derivative at $x = 0$, that is $f'(0)$. We can use the limit definition of the derivative to write the following

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}.$$

Compute this limit. If it does not exist justify why. Draw the graph of $f(x) = |x|$. What happens when you try to draw a tangent line at $x = 0$. What can you conclude about $f'(0)$ after observing your limit computation?

Solution: Let us try to compute $f'(0)$ where $f(x) = |x|$ using the definition of the derivative. We have

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{|0+h| - |0|}{h} \\ &= \lim_{h \to 0} \frac{|h| - 0}{h} \\ &= \lim_{h \to 0} \frac{|h|}{h}. \end{aligned}$$

To compute this limit we consider the one sided limits. Indeed, $|h|$ behaves differently depending on whether $h > 0$ or $h < 0$. Note

$$|h| = \begin{cases} h & , h > 0 \\ -h & , h < 0 \end{cases}.$$

Now as h approaches from above (right) we have

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

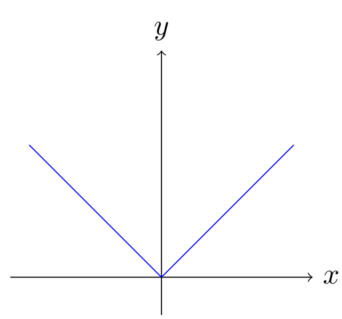
and now from below (left)

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1.$$

Therefore, since the two one sided limits do not match, we conclude that

$$f'(0) = \lim_{h \to 0} \frac{|h|}{h}$$

does not exist. Our function has no derivative at $x = 0$ and we say it is not differentiable at $x = 0$. Our limit computation can be corroborated by the graph of $f(x) = |x|$.



As hard as we try it is impossible to draw a proper tangent line at $x = 0$; the function comes to a sharp corner. We notice that for $x > 0$ the slope of the tangent line is 1 and for $x < 0$ the slope of the tangent line is -1. We somehow want to have two tangent lines, one from the left and one from the right. However, tangent lines must be unique and in this case the derivative does not exist.