Math 190 Homework 8: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:

1. A spherical snow ball is melting such that its surface area is decreasing at a rate of $0.5\text{cm}^2/\text{min}$. How fast is the volume decreasing when the radius is 6cm? the Volume and Surface Area of a sphere is given by

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad A = 4\pi r^2$$

respectively.

Solution: We have dA/dt and we seek dV/dt. If we differentiate both sides of the volume equation with respect to time we see

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

If we can find now dr/dt we should be able uncover the desired rate. We now use the area equation to solve for dr/dt. Differentiating both sides in t yields

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$

and so

$$\frac{dr}{dt} = \frac{1}{8\pi r}\frac{dA}{dt}.$$

With dr/dt in hand we can finish the problem. We find

$$\begin{aligned}\frac{dV}{dt} &= 4\pi r^2 \frac{1}{8\pi r}\frac{dA}{dt} \\ &= \frac{r}{2}\frac{dA}{dt}\end{aligned}$$

and so when $r = 6$ cm we have

$$\begin{aligned}\frac{dV}{dt} &= \frac{1}{2}(6\text{cm})(-0.5\text{cm}^2/\text{min}) \\ &= -\frac{6}{4}\text{cm}^3/\text{min} \\ &= -\frac{3}{2}\text{cm}^3/\text{min}.\end{aligned}$$

The volume of the snow ball is decreasing at a rate of $3/2\ \text{cm}^3/\text{min}$.

AltSol: Alternatively we can just get rid of r in favour of A at the start and avoid differentiating twice. Solving for r in the area equation yields

$$r = \left(\frac{A}{4\pi}\right)^{1/2}.$$

We can now substitute this to the volume equation to achieve V in terms of A; that is

$$V = \frac{4}{3}\pi\frac{A^{3/2}}{(4\pi)^{3/2}}.$$

At this point we differentiate both sides of the equation with respect to t

$$\frac{dV}{dt} = \frac{4}{3}\frac{3}{2}\pi\frac{1}{(4\pi)^{3/2}}A^{1/2}\frac{dA}{dt}.$$

Note that $A^{1/2} = (4\pi)^{1/2}r$ after observing the area equation. Substituting this produces

$$\begin{aligned}\frac{dV}{dt} &= \frac{1}{2}\frac{(4\pi)}{(4\pi)^{3/2}}(4\pi)^{1/2}r\frac{dA}{dt}\\ \frac{dV}{dt} &= \frac{1}{2}r\frac{dA}{dt}\\ &= \frac{6}{4} = \frac{3}{2}\end{aligned}$$

as before.

2. A boat is travelling down a river along the curve

$$y = \sqrt{x+4}.$$

You notice that the y-coordinate of its position is decreasing at a rate of 10 knots when the boat is at the point $(-1/2, \sqrt{7/2})$. How fast is the x-coordinate of the boat changing at this point in time? Which is changing faster: the x-coordinate or the y-coordinate? Explain why you would expect this at the start of the problem, perhaps in reference to your picture.

Solution: We have dy/dt and we would like to find dx/dt. We have already an equation relating x and y. Let us differentiate both sides with respect to t. We see

$$\frac{dy}{dt} = \frac{1}{2}(x+4)^{-1/2}\frac{dx}{dt}$$

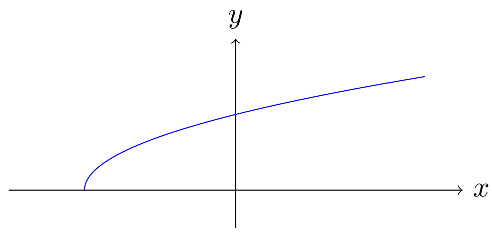
and solving for dx/dt gives

$$\frac{dx}{dt} = 2\sqrt{x+4}\frac{dy}{dt}.$$

We can substitute our known values to find

$$\begin{aligned}\frac{dx}{dt} &= 2\sqrt{-1/2+4}(-10 \text{ knots})\\ &= -20\sqrt{7/2} \text{ knots}.\end{aligned}$$

We see now that the x-coordinate is changing faster than the y-coordinate. We expect this based on the shape of the square root function. Observing the graph we see that the x-coordinate is changing faster than the y-coordinate. Over some distance the function will only change a little in y but changes dramatically in x.

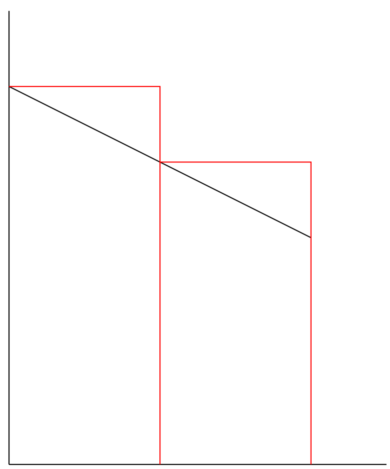


3. Consider the function

$$f(x) = -\frac{1}{2}x + 5$$

   (a) Approximate the area under the curve on the interval $[0, 4]$ using Riemann Sums. Use left endpoints and two bars ($n = 2$).
   (b) Now approximate the same area using four bars ($n = 4$), again with left endpoints.
   (c) Compute the exact area, either by integrating or by drawing a picture and using area formulas. Which approximation is better? Are your approximations over or under estimates? Explain why you would expect this at the start of the problem, perhaps in reference to your picture.

Solution: (a) We first approximate the area using two bars. See the figure bellow. We can



represent this sum in the following way

$$\sum_{i=0}^{1} f(x_i)\Delta x$$

where Δx is the width of the bars and the x_i are the left endpoints. We have

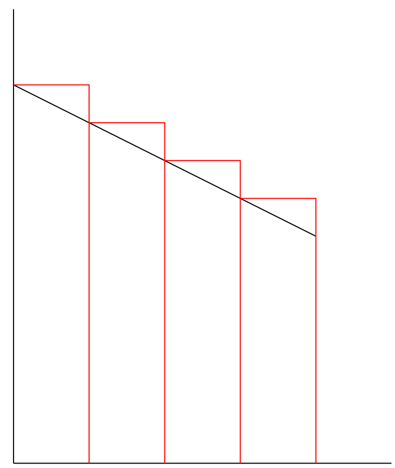
$$\Delta x = \frac{b-a}{n} = \frac{4-0}{2} = 2$$

and $x_0 = 0$ and $x_1 = 2$. In this way we approximate the integral as follows

$$\begin{aligned}\sum_{i=0}^{1} f(x_i)\Delta x &= f(x_0)\Delta x + f(x_1)\Delta x \\ &= 5 \cdot 2 + 4 \cdot 2 \\ &= 18.\end{aligned}$$

So our approximation of the area is 18.

(b) Now we approximate the area using four bars. Observe again the figure. We can represent



this sum in the following way

$$\sum_{i=0}^{3} f(x_i)\Delta x$$

where $\Delta x = (4-0)/4 = 1$ and $x_i = i$. We compute

$$\begin{aligned}\sum_{i=0}^{3} f(x_i)\Delta x &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 5 \cdot 1 + 4.5 \cdot 1 + 4 \cdot 1 + 3.5 \cdot 1 \\ &= 17\end{aligned}$$

and so find 17 as our approximation.

(c) Let us compute the exact area. This can be done by splitting the area under the curve into a triangle and a rectangle. In this way

$$\int_0^4 \left(-\frac{1}{2}x + 5\right) dx = 4 \cdot 3 + \frac{2 \cdot 4}{2} = 12 + 4 = 16.$$

We can also compute the integral directly using the Fundamental Theorem of Calculus

$$\int_0^4 \left(-\frac{1}{2}x + 5\right) dx = -\frac{1}{4}x^2 + 5x\Big|_{x=0}^{x=4} = -\frac{1}{4}4^2 + 5 \cdot 4 = 16.$$

In light of the above we achieve the exact area, 16. We notice that the second approximation is better, having used more bars our computed area is closer to the exact area. Both estimates are overestimates. This is to be expected after observing each graph since we see our bars are capturing additional area beyond the desired region.

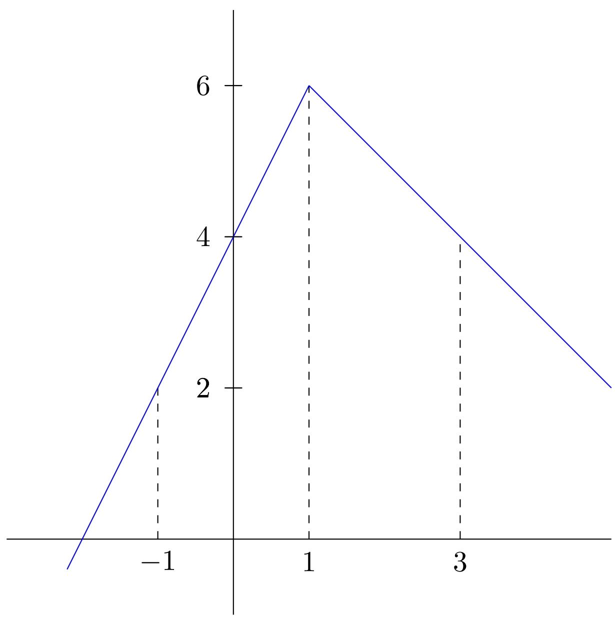
4. Consider the function

$$g(x) = \begin{cases} 2(x-1)+6, & x \leq 1 \\ -x+7, & x > 1 \end{cases}.$$

   (a) Sketch the graph of $g(x)$.
   (b) Compute

$$\int_{-1}^{3} g(x)dx.$$

Solution: First, we sketch a graph of $g(x)$.



We seek the area under this curve between $x = -1$ and $x = 3$. We will split the integral into two parts, these being the area under the curve between $x = -1$ and $x = 1$ and then the area under the curve between $x = 1$ and $x = 3$. That is we compute

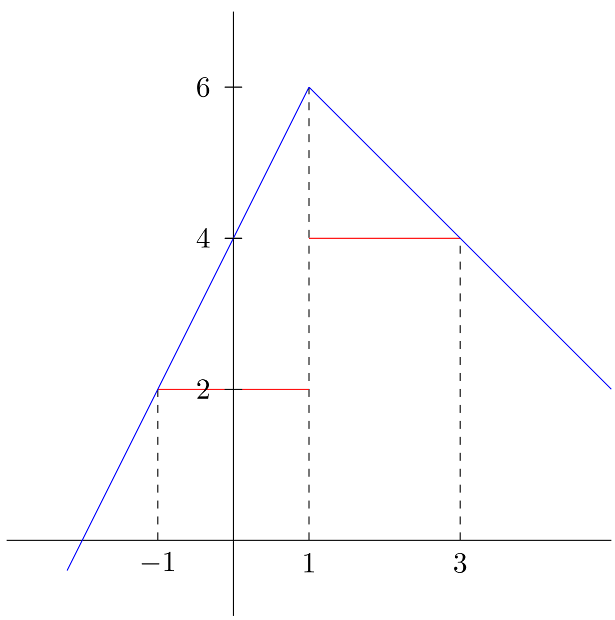
$$\int_{-1}^{3} g(x)dx = \int_{-1}^{1} g(x)dx + \int_{1}^{3} g(x)dx.$$

We know that between $x = -1$ and $x = 1$ the function $g(x)$ takes the first branch and so we substitute this line into the first integral. We similarly compute the second integral. Observe

$$\begin{aligned}
\int_{-1}^{3} g(x)dx &= \int_{-1}^{1} g(x)dx + \int_{1}^{3} g(x)dx \\
&= \int_{-1}^{1} (2(x-1)+6)\,dx + \int_{1}^{3} (-x+7)\,dx \\
&= \int_{-1}^{1} (2x+4)dx + \int_{1}^{3} (-x+7)\,dx \\
&= \Big[x^2+4x\Big]_{-1}^{1} + \Big[-\frac{x^2}{2}+7x\Big]_{1}^{3} \\
&= 1+4-(1-4)-\frac{9}{2}+7\cdot 3-\left(-\frac{1}{2}+7\right) \\
&= 8-4+14 \\
&= 18.
\end{aligned}$$

Alternatively we could split the total area into two rectangles and two triangles to achieve the same result. That is

$$\begin{aligned}
\int_{-1}^{3} g(x)dx &= 2\cdot 2+\frac{2\cdot 4}{2}+2\cdot 4+\frac{2\cdot 2}{2} \\
&= 4+4+8 \\
&= 18.
\end{aligned}$$



5. Compute the following definite integral

$$\int_{0}^{\pi} \left(\cos x + x^3 + 4\right) dx.$$

Solution:
We compute using the Fundamental Theorem of Calculus

$$\begin{aligned}\int_0^{\pi} \left(\cos x + x^3 + 4\right) dx &= \sin x + \frac{1}{4}x^4 + 4x\Big|_0^{\pi} \\ &= \sin(\pi) + \frac{\pi^4}{4} + 4\pi - \left(\sin(0) + \frac{0^4}{4} + 4(0)\right) \\ &= \frac{\pi^4}{4} + 4\pi.\end{aligned}$$