Math 190 Homework 9: Solution

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:

1. If we know that
   - $\int_{-1}^{3} f(x)dx = 6$
   - $\int_{-1}^{3} g(x)dx = -3$

   then compute

   (a) $\int_{-1}^{3} \big(2f(x) - g(x)\big)dx$

   (b) $\int_{-1}^{3} \big(4f(x) + 5g(x)\big)dx$

   Solution: (a) We split the integral in order to use our known information. Observe

$$\begin{aligned}\int_{-1}^{3} \big(2f(x) - g(x)\big)dx &= \int_{-1}^{3} 2f(x) + \int_{-1}^{3} \big(-g(x)\big)dx \\ &= 2\int_{-1}^{3} f(x) - \int_{-1}^{3} g(x)dx \\ &= 2(6) - (-3) \\ &= 12 + 3 \\ &= 15.\end{aligned}$$

   (b) We apply the same manipulation again to see

$$\begin{aligned}\int_{-1}^{3} \big(4f(x) + 5g(x)\big)dx &= \int_{-1}^{3} 4f(x) + \int_{-1}^{3} 5g(x)dx \\ &= 4\int_{-1}^{3} f(x) + 5\int_{-1}^{3} g(x)dx \\ &= 4(6) + 5(-3) \\ &= 24 - 15 \\ &= 9.\end{aligned}$$

2. Find a function $F(x)$ such that $F'(x) = 5\cos x - \sqrt{3}\sin x$ and $F(\pi/6) = 4$.

Solution: We seek $F(x)$ where $F'(x) = 5\cos x - \sqrt{3}\sin x$ and so we integrate to see

$$F(x) = 5\sin x + \sqrt{3}\cos x + C.$$

We now wish to ensure that $F(\pi/6) = 4$. We therefore select C to satisfy this condition. That is

$$\begin{aligned}
4 &= F\left(\frac{\pi}{6}\right) \\
&= 5\sin\left(\frac{\pi}{6}\right) + \sqrt{3}\cos\left(\frac{\pi}{6}\right) + C \\
&= 5\left(\frac{1}{2}\right) + \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) + C \\
&= \frac{5}{2} + \frac{3}{2} + C \\
&= \frac{8}{2} + C \\
&= 4 + C.
\end{aligned}$$

Hence, $C = 4 - 4 = 0$ and so our desired $F(x)$ is

$$F(x) = 5\sin x + \sqrt{3}\cos x.$$

3. Compute the following indefinite integrals
   (a) $\int 2dx$
   (b) $\int \sin(3x)dx$
   (c) $\int e^{x+1}dx$

Solution: (a)

$$\int 2dx = 2x + C$$

(b) With a little guess work we can find the form of the integral

$$\int \sin(3x)dx = -\frac{1}{3}\cos(3x) + C.$$

If you don't want to guess and check we can solve this by making a substitution. Take $u = 3x$ then $du = 3dx$ and we have

$$\int \sin(3x)dx = \frac{1}{3}\int \sin u du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(3x) + C.$$

(c) We compute

$$\int e^{x+1}dx = e^{x+1} + C.$$

Again, if you prefer not to guess we can use a simple substitution. Let $u = x+1$ so that $du = dx$ and now

$$\int e^{x+1}dx = \int e^u du = e^u + C = e^{x+1} + C.$$

4. Compute the following indefinite integral

$$\int \frac{\sqrt{x} - 2x^4}{\sqrt{x^3}} dx.$$

Solution: Let us first modify the integrand until the form is pleasing to us. We split the fraction and combine powers of x

$$\begin{aligned}\int \frac{\sqrt{x} - 2x^4}{\sqrt{x^3}} dx &= \int \left(\frac{x^{1/2}}{x^{3/2}} - \frac{2x^4}{x^{3/2}} \right) dx \\ &= \int \left(x^{-1} - 2x^{5/2} \right) dx \\ &= \ln|x| - 2\frac{2}{7}x^{7/2} + C \\ &= \ln|x| - \frac{4}{7}x^{7/2} + C.\end{aligned}$$

5. (a) Suppose that

$$\int_{-1}^{1} f(x)dx = -2 \quad \text{and} \quad \int_{1}^{4} f(x)dx = -3.$$

Find

$$\int_{-1}^{4} f(x)dx.$$

Explain how you know using a picture.

(b) What is

$$\int_{2}^{2} f(x)dx?$$

Explain how you know using a picture.

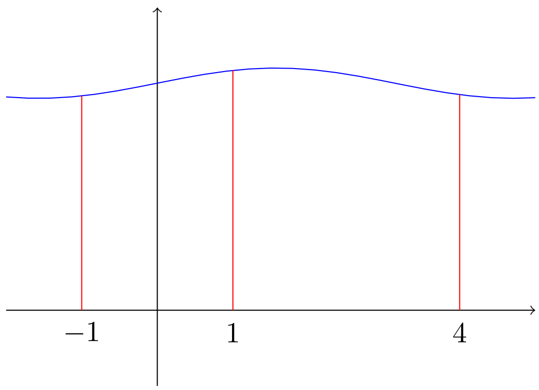
Solution: (a) We claim that

$$\int_{-1}^{1} f(x)dx + \int_{1}^{4} f(x)dx = \int_{-1}^{4} f(x)dx.$$

In this way

$$\int_{-1}^{4} f(x)dx = -2 - 3 = -5.$$

To address the claim observe the following figure:



We see that the area under the curve between -1 and 4 can be broken into two areas, the first being between -1 and 1 and the second between 1 and 4. In fact, in general we have

$$\int_{a}^{b} f(x)dx + \int_{b}^{c} f(x)dx = \int_{a}^{c} f(x)dx.$$

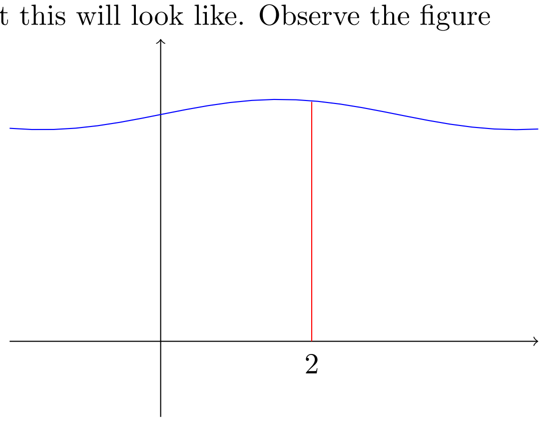
We can also think about this identity in terms of the Fundamental Theorem of Calculus. Suppose we have an anti-derivative of $f(x)$ and call it $F(x)$. Then

$$\int_{a}^{b} f(x)dx + \int_{b}^{c} f(x)dx = F(b) - F(a) + F(c) - F(b) = F(c) - F(a) = \int_{a}^{c} f(x)dx$$

as required.
(b) Consider

$$\int_{2}^{2} f(x)dx.$$

Let us think about what this will look like. Observe the figure



The region represented by this integral is nothing but a one dimensional line. This one dimensional object has no area and so

$$\int_2^2 f(x)dx = 0.$$

In general we have that

$$\int_a^b f(x)dx = 0.$$

Again, we can think about this property in terms of the FTofC. For anti-derivative $F(x)$ we see

$$\int_2^2 f(x)dx = F(2) - F(2) = 0$$

as desired.