

Final Exam Duration: 2.5 hours

This test has 10 questions on 12 pages, for a total of 75 points.

- Read all the questions carefully before starting to work.
- All questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: Matt Last Name: Coles

Student-No: _____

Signature: _____

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	8	8	5	5	15	5	9	7	8	5	75
Score:											

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Justify your answers and show all your work. Unless otherwise indicated simplification of answers is not required.

1. Consider the following function

$$f(x) = -2x + 4\sqrt{x} - \frac{1}{x^2} + \pi.$$

4 marks

(a) Compute the derivative of the function $f(x)$.

$$f(x) = -2x + 4x^{1/2} - x^{-2} + \pi$$

$$\begin{aligned} f'(x) &= -2 + \frac{4}{2}x^{-1/2} + 2x^{-3} + 0 \\ &= -2 + 2x^{-1/2} + 2x^{-3} \end{aligned}$$

4 marks

(b) Compute the general anti-derivative of the function $f(x)$.

Let $F(x)$ be such that $F'(x) = f(x)$

Then,

$$F(x) = -x^2 + 4 \cdot \frac{2}{3} x^{3/2} + x^{-1} + \pi x + C.$$

4 marks 2. (a) Compute

$$\int_1^2 \frac{2-x}{\sqrt{x}} dx.$$

$$\begin{aligned} \int_1^2 \frac{2-x}{\sqrt{x}} dx &= \int_1^2 \left(\frac{2}{\sqrt{x}} - \frac{x}{\sqrt{x}} \right) dx = \int_1^2 \{ 2x^{-1/2} - x^{1/2} \} dx \\ &= 4x^{1/2} - \frac{2}{3}x^{3/2} \Big|_{x=1}^2 \\ &= 4\sqrt{2} - \frac{2}{3}(2)^{3/2} - \left(4 - \frac{2}{3} \right) \\ &= 4\sqrt{2} - \frac{2}{3}(2)^{3/2} - 4 + \frac{2}{3}. \end{aligned}$$

4 marks (b) Find a function $g(x)$ satisfying $g(\pi) = 1$ such that

$$g'(x) = \sin x + \cos(4x) + e^x.$$

$$g(x) = -\cos x + \frac{1}{4} \sin(4x) + e^x + C.$$

Set,

$$\begin{aligned} 1 = g(\pi) &= -\cos(\pi) + \frac{1}{4} \sin(4\pi) + e^\pi + C. \\ &= 1 + 0 + e^\pi + C. \end{aligned}$$

$$\Rightarrow 1 = 1 + e^\pi + C$$

$$C = -e^\pi$$

$$\Rightarrow g(x) = -\cos x + \frac{1}{4} \sin(4x) + e^x - e^\pi.$$

5 marks 3. Find the equation of the tangent line to

$$f(x) = \frac{\cos(2x)}{x}$$

at the point $x = \pi$.

Apply QR:

$$f'(x) = \frac{-\sin(2x) \cdot \frac{1}{2}x - \cos(2x)}{x^2}$$

$$f'(\pi) = \frac{-\frac{\pi}{2}\sin(2\pi) - \cos(2\pi)}{\pi^2}$$

$$= -\frac{1}{\pi^2} \leftarrow \text{slope of the tangent.}$$

Use slope-point form

$$y - y_1 = m(x - x_1)$$

$$m = -\frac{1}{\pi^2}$$

$$x_1 = \pi$$

$$y_1 = f(\pi) = \frac{\cos(2\pi)}{\pi} = \frac{1}{\pi}$$

So,

$$y - \frac{1}{\pi} = -\frac{1}{\pi^2}(x - \pi)$$

is the equation of the desired tangent line.

5 marks 4. Find the derivative of

$$f(x) = xe^{2x} \sin(x^2).$$

We apply product rule twice.
(Alternatively, use triple product rule)

$$\begin{aligned} f'(x) &= (x)'e^{2x} \sin(x^2) + x(e^{2x} \sin(x^2))' \\ &= e^{2x} \sin(x^2) + x(2e^{2x} \sin(x^2) + e^{2x} \cdot 2x \cos(x^2)) \\ &= e^{2x} \sin(x^2) + 2xe^{2x} \sin(x^2) + 2xe^{2x} \cos(x^2). \end{aligned}$$

5. Compute the following integrals.

5 marks

(a)

$$\int \frac{e^x}{(e^x + 1)^2} dx$$

$$\text{Let } u = e^x + 1 \quad du = e^x dx$$

$$\int \frac{1}{(e^x + 1)^2} e^x dx = \int \frac{1}{u^2} du = -\frac{1}{u} + C$$

$$= \frac{-1}{e^x + 1} + C.$$

5 marks

(b)

$$\int_0^{\pi/2} \sin x \cos x dx$$

when $x = 0$ $u = 0$ when $x = \pi/2$ $u = 1$.

$$\begin{aligned} \text{Let } u = \sin x, \quad du = \cos x dx \\ \int_{x=0}^{x=\pi/2} \sin x \cos x dx &= \int_{u=0}^{u=1} u du = \frac{1}{2} u^2 \Big|_{u=0}^{u=1} \\ &= \frac{1}{2}. \end{aligned}$$

5 marks

(c)

$$\int x^2 \ln x \, dx$$

Let $u = \ln x$ and $dv = x^2 dx$

then $du = \frac{1}{x} dx$ and $v = \frac{x^3}{3}$.

$$\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 \, dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + C.$$

$$= \frac{x^3}{3} \ln x - \frac{1}{9} x^3 + C.$$

5 marks

6. A spherical snow ball is melting such that its surface area is decreasing at a rate of $0.5 \text{ cm}^2/\text{min}$. How fast is the volume decreasing when the radius is 6 cm ? The Volume and Surface Area of a sphere are given by

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad A = 4\pi r^2$$

respectively.

We seek $\frac{dV}{dt}$ and have $\frac{dA}{dt}$.
We first, however, find $\frac{dr}{dt}$.

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{8\pi r} \frac{dA}{dt}$$

$$\text{Now, } \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$= 4\pi r^2 \frac{1}{8\pi r} \frac{dA}{dt}$$

$$= \frac{r}{2} \frac{dA}{dt}$$

Want $\frac{dV}{dt}$

$$\text{@ } r = 6 \text{ cm, } \frac{dA}{dt} = 1/2 \text{ cm}^2/\text{min.}$$

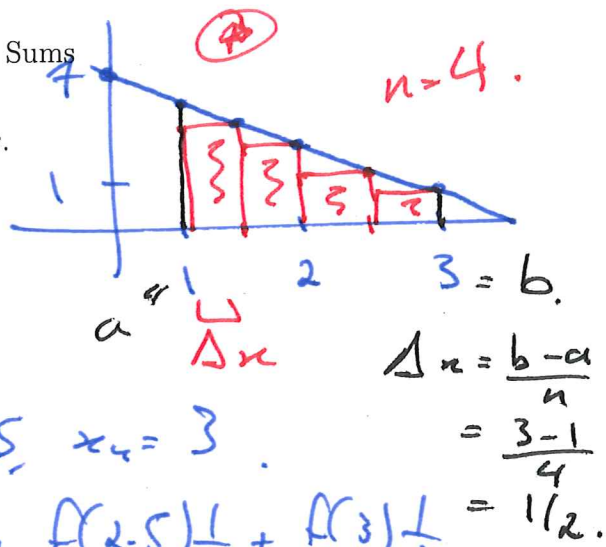
$$\begin{aligned} \frac{dV}{dt} &= \frac{6 \text{ cm}}{2} \cdot \frac{1}{2} \text{ cm}^2/\text{min} \\ &= \frac{3}{2} \text{ cm}^3/\text{min} \end{aligned}$$

4 marks

7. (a) Approximate the following integral using Riemann Sums

$$\int_1^3 (-2x + 7) dx.$$

Use right endpoints and $n = 4$ (ie. four bars).



$$\int_1^3 (-2x + 7) dx \approx \sum_{i=1}^4 f(x_i) \Delta x$$

$$x_1 = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3.$$

$$\sum_{i=1}^4 f(x_i) \Delta x = f(1.5) \cdot \frac{1}{2} + f(2) \cdot \frac{1}{2} + f(2.5) \cdot \frac{1}{2} + f(3) \cdot \frac{1}{2} = 1/2.$$

$$= 4 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}$$

$$= 2 + \frac{3}{2} + 1 + \frac{1}{2}$$

$$= 3 + 2 = 5.$$

2 marks

(b) Is your approximation less than, greater than, or exactly equal to the true value of the integral? Explain why.

Less than. Observe the graph. Since the function is decreasing and we used right endpoints we have missed some area.

3 marks

(c) Sketch the graph of a new function where an approximation with Riemann Sums is exactly equal to the area under the curve.

Consider the function $f(x) = 4$. The constant function.

No matter how many bars we use our 'approximation' with Riemann Sums is exactly the area under the curve.

7 marks

8. The rate of change of the height of an elevator is give by

$$r(t) = te^t$$

in meters/second. If at $t = 0$ seconds the elevator is 1 meter off the ground, how high is the elevator after 2 seconds have passed?

We are interested in the height and have its rate of change.

$$\begin{aligned} h(t) &= \int r(t) dt \\ &= \int te^t dt. \end{aligned}$$

$$\begin{aligned} \text{Let } u &= t & du &= dt \\ & & & & dv &= e^t dt \\ & & & & v &= e^t \end{aligned}$$

$$\begin{aligned} &= te^t - \int e^t dt \\ &= te^t - e^t + C. \end{aligned}$$

We know, $h(0) = 1$ so

$$\begin{aligned} 1 = h(0) &= 0 \cdot e^0 - e^0 + C \\ &= -1 + C \end{aligned}$$

$$\Rightarrow C = 2.$$

Hence, $h(t) = te^t - e^t + 2.$

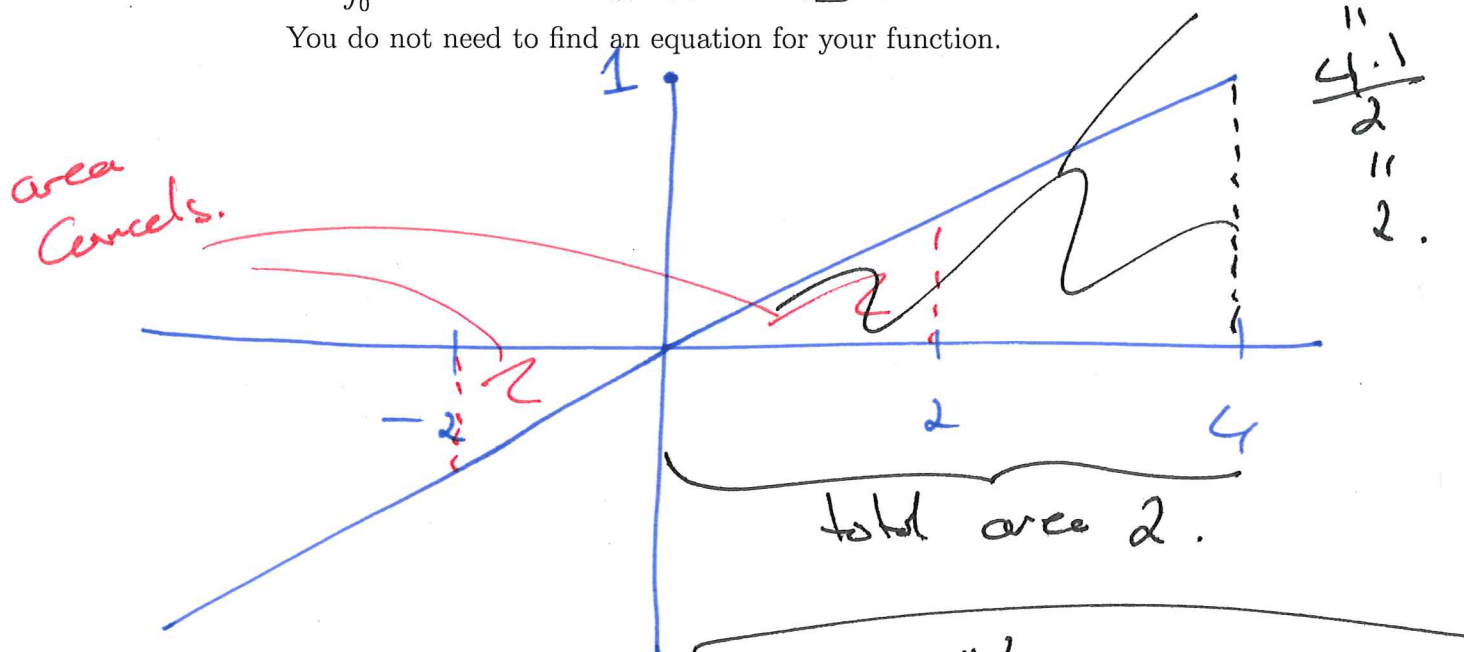
After 2 seconds we have,

$$\begin{aligned} h(2) &= 2e^2 - e^2 + 2 \text{ meters high.} \\ &= e^2 + 2. \end{aligned}$$

4 marks 9. (a) Sketch the graph of a function $f(x)$ satisfying the following two properties:

- $\int_{-2}^2 f(x) dx = 0$
 - $\int_0^4 f(x) dx = 2$
- There are many examples.
One is:

You do not need to find an equation for your function.



4 marks (b) Find two values of b such that

Solution #1

Ensure you justify your answer fully.

Compute:

$$\int_{-\pi}^b \sin(2x) dx = -\frac{1}{2} \cos(2x) \Big|_{x=-\pi}^{x=b}$$

$$= -\frac{1}{2} \cos(2b) + \frac{1}{2} \cos(-2\pi)$$

$$= \frac{1}{2} - \frac{1}{2} \cos(2b)$$

Need $\cos(2b) = 1$

Take, $b = 0$ and $b = \pi$.

In general

$$2b = 2\pi n$$

$$b = \pi n, \quad n \in \mathbb{Z}$$

Solution #2

$$\int_{-\pi}^b \sin(2x) dx = 0$$

First of all

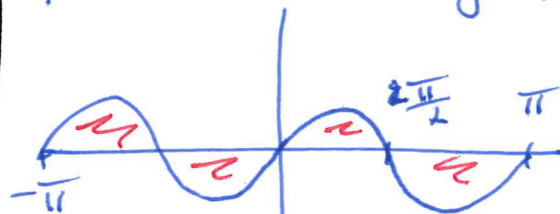
$$\int_{-\pi}^{\pi} \sin(2x) dx = 0$$

So, $b = -\pi$ works.

Second, $\sin(2x)$ is odd so $\int_{-\pi}^{\pi} \sin(2x) dx = 0$

$$b = \pi$$

Third draw a graph.



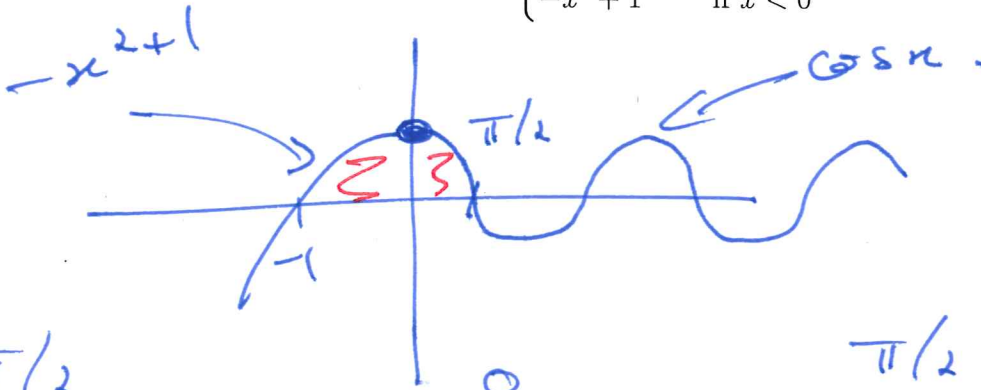
$b = 0$ or $b = \pi$ both work from the graph.

5 marks 10. Compute the integral

$$\int_{-1}^{\pi/2} f(x) dx$$

where

$$f(x) = \begin{cases} \cos x & \text{if } x \geq 0 \\ -x^2 + 1 & \text{if } x < 0 \end{cases}$$



$$\int_{-1}^{\pi/2} f(x) dx = \int_{-1}^0 f(x) dx + \int_0^{\pi/2} f(x) dx$$

$$= \int_{-1}^0 (-x^2 + 1) dx + \int_0^{\pi/2} \cos x dx$$

$$= \left. -\frac{x^3}{3} + x \right|_{-1}^0 + \left. \sin x \right|_0^{\pi/2}$$

$$= \frac{-0^3}{3} + 0 - \left(\frac{-1^3}{3} + 1 \right) + \sin\left(\frac{\pi}{2}\right) + \sin(0)$$

$$= -\frac{1}{3} + 1 + 1$$

$$= -\frac{1}{3} + 2 = \frac{5}{3}$$