Midterm Exam Duration: 45 minutes

This test has ***5 questions*** *on* ***6 pages****, for a total of 35 points.*

- Read all the questions carefully before starting to work.
- All questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ______________________ Last Name: ______________________

Student-No: ____

Signature: ____

Question:	1	2	3	4	5	Total
Points:	8	8	8	5	6	35
Score:						

Justify your answers and show all your work. Unless otherwise indicated simplification of answers is not required.

1. Compute the derivatives of the following functions

4 marks (a)

$$f(x) = \left(x^{5/2}\right)^2 + 5e^e - e^x + \frac{4}{\sqrt[3]{x^2}}$$

Solution: First we write our function as the following

$$f(x) = x^5 + 5e^e - e^x + 4x^{-2/3}$$

and then take the derivative using our power and exponential rules

$$f'(x) = 5x^4 - e^x - \frac{8}{3}x^{-5/3}$$

If you didn't notice the simplification of the first or last term (simplification is your friend *before* you take the derivative) you can use chain or quotient rule or a combination thereof.

4 marks (b)

$$g(x) = x^3 \ln(x^2)$$

Solution: We apply chain rule within product rule:

$$\begin{aligned} g'(x) &= 3x^2 \ln x^2 + x^3 \frac{1}{x^2} 2x \\ &= 6x^2 \ln x + 2x^2 \end{aligned}$$

where simplification is not necessary. Alternatively we can simplify before computing and see

$$g(x) = 2x^3 \ln x$$

and now

$$\begin{aligned} g'(x) &= 6x^2 \ln x + 2x^3 \frac{1}{x} \\ &= 6x^2 \ln x + 2x^2. \end{aligned}$$

1 mark 2. (a) Evaluate the following two quantities

$$\sin(\pi) \text{ and } \cos(\pi)$$

Solution: We compute using the unit circle or drawing the graph of each function

$$\sin\pi = 0$$
$$\cos\pi = -1$$

7 marks (b) Find the equation of the tangent line to

$$f(x) = \frac{e^{\sin x}}{\cos x}$$

at the point $x = \pi$.

Solution: We start by finding the slope of the tangent line. For this we need the derivative. Apply now quotient rule to see

$$f'(x) = \frac{\cos x e^{\sin x}\cos x + e^{\sin x}\sin x}{\cos^2 x}$$
$$f'(x) = \frac{e^{\sin x}\cos^2 x + e^{\sin x}\sin x}{\cos^2 x}$$

where we have used a chain rule as well. To find the slope of the tangent line at $x = \pi$ we compute

$$f'(\pi) = \frac{e^0(-1)^2 + e^0(0)}{(-1)^2} = 1$$

Now with the slope in hand our equation for the tangent line takes the form (using point-slope form)

$$y - y_1 = m(x - x_1)$$
$$y - y_1 = 1(x - x_1).$$

We have the point $x_1 = \pi$ and so we find

$$y_1 = f(\pi) = \frac{e^0}{-1} = -1$$

All together the equation of the desired tangent line is

$$y + 1 = x - \pi$$

3. For this problem show all relevant limit computations.

3 marks (a) Matt's favourite function is

$$f(x) = \frac{2}{e^x + e^{-x}}.$$

Find the equation(s) of all horizontal asymptotes of this function.

Solution: To find the H.A. consider the following limits noting that $e^x \to \infty$ as $x \to \infty$ and that $e^x \to 0$ as $x \to -\infty$

$$\lim_{x\to\infty} \frac{2}{e^x + e^{-x}} = \frac{2}{\text{“}\infty\text{”} + 0} = 0$$

and

$$\lim_{x\to-\infty} \frac{2}{e^x + e^{-x}} = \frac{2}{0 + \text{“}\infty\text{”}} = 0.$$

And so we have a H.A. at $y = 0$.

5 marks (b) Consider now the function

$$g(x) = \frac{\ln x}{x-2}.$$

Find the equation(s) of all vertical asymptotes of this function. Ensure you determine whether the relevant one sided limits equal $+\infty$ or $-\infty$.

Solution: We first identify the two candidates $x = 2$ and $x = 0$. We suspect $x = 2$ since this is where the denominator is zero. Drawing the graph of $\ln x$ tips us off that there might be an asymptote at $x = 0$. We note that as $x \to 0^+$, $\ln x \to -\infty$. We now compute the relevant one sided limits. First around $x = 2$:

$$\lim_{x\to 2^+} \frac{\ln x}{x-2} = \frac{\ln(2)}{0^+} = \infty$$

and

$$\lim_{x\to 2^-} \frac{\ln x}{x-2} = \frac{\ln(2)}{0^-} = -\infty$$

noting that $\ln(2) > 0$. Now we investigate $x = 0$. We consider only the limit as $x \to 0^+$ since $g(x)$ is not defined for negative values of x.

$$\lim_{x\to 0^+} \frac{\ln x}{x-2} \frac{\text{“}-\infty\text{”}}{-2} = \infty.$$

And so we have two V.A. One at $x = 2$ and one at $x = 0$.

1 mark 4. (a) Given function $f(x)$ state the definition of its derivative.

Solution: The definition of the derivative is given by the following limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

4 marks (b) Using the above definition compute the derivative of

$$f(x) = \frac{1}{4x}.$$

No credit will be given for using a method other than the definition of the derivative.

Solution: We compute

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{1}{h}\big(f(x+h)-f(x)\big) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{4(x+h)} - \frac{1}{4x}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x}{4(x+h)x} - \frac{(x+h)}{4(x+h)x}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x-x-h}{4(x+h)x}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{4(x+h)x}\right) \\
&= \lim_{h\to 0} \frac{-1}{4(x+h)x} \\
&= \frac{-1}{4(x+0)x} \\
&= -\frac{1}{4x^2}
\end{aligned}$$

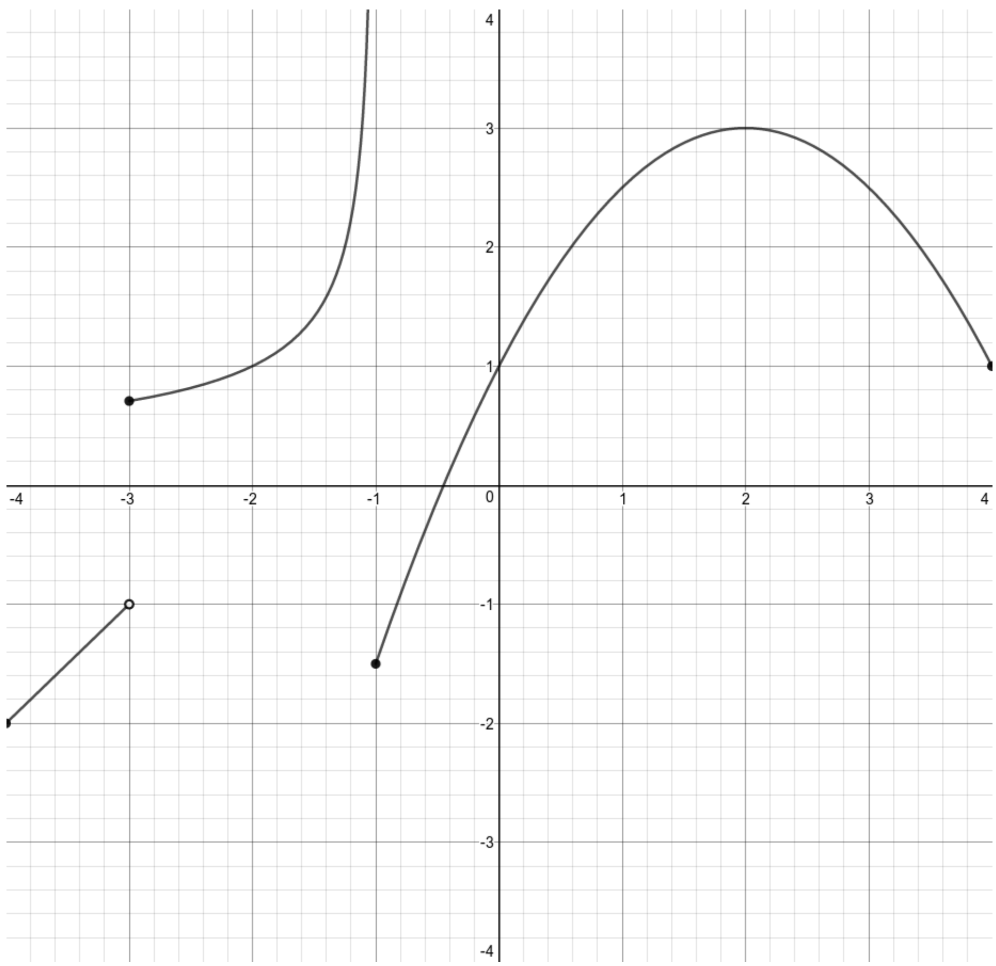
which we could confirm with product rule.

6 marks 5. Sketch the graph of a function satisfying the following properties:

- The domain of $f(x)$ is $\{x \in \mathbb{R} : -4 \leq x \leq 4\}$
- $f(x)$ has a vertical asymptote at $x = -1$
- $\lim_{x \to -3} f(x)$ does not exist
- $f'(2) = 0$
- $f'(1) > 0$
- $f'(3) < 0$

You do not need to find an equation for your function. Use the axes below.

Solution: There are many graphs that will satisfy the above six conditions. Here is an example of one such graph.



Observe that the slope is positive at $x = 1$, zero at $x = 2$, and negative at $x = 3$. Notice that $f(x)$ is defined for all $x \in [-4, 4]$ including at $x = -1$.