

Midterm Exam Duration: 45 minutes*This test has 5 questions on 6 pages, for a total of 35 points.*

- Read all the questions carefully before starting to work.
- All questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: _____ Last Name: _____

Student-No: _____

Signature: _____

Question:	1	2	3	4	5	Total
Points:	8	8	8	5	6	35
Score:						

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Justify your answers and show all your work. Unless otherwise indicated simplification of answers is not required.

1. Compute the derivatives of the following functions

4 marks

(a)

$$f(x) = (x^{5/2})^2 + 5e^e - e^x + \frac{4}{\sqrt[3]{x^2}}$$

Solution: First we write our function as the following

$$f(x) = x^5 + 5e^e - e^x + 4x^{-2/3}$$

and then take the derivative using our power and exponential rules

$$f'(x) = 5x^4 - e^x - \frac{8}{3}x^{-5/3}$$

If you didn't notice the simplification of the first or last term (simplification is your friend *before* you take the derivative) you can use chain or quotient rule or a combination thereof.

4 marks

(b)

$$g(x) = x^3 \ln(x^2)$$

Solution: We apply chain rule within product rule:

$$\begin{aligned} g'(x) &= 3x^2 \ln x^2 + x^3 \frac{1}{x^2} 2x \\ &= 6x^2 \ln x + 2x^2 \end{aligned}$$

where simplification is not necessary. Alternatively we can simplify before computing and see

$$g(x) = 2x^3 \ln x$$

and now

$$\begin{aligned} g'(x) &= 6x^2 \ln x + 2x^3 \frac{1}{x} \\ &= 6x^2 \ln x + 2x^2. \end{aligned}$$

- 1 mark 2. (a) Evaluate the following two quantities

$$\sin(\pi) \text{ and } \cos(\pi)$$

Solution: We compute using the unit circle or drawing the graph of each function

$$\begin{aligned}\sin \pi &= 0 \\ \cos \pi &= -1\end{aligned}$$

- 7 marks (b) Find the equation of the tangent line to

$$f(x) = \frac{e^{\sin x}}{\cos x}$$

at the point $x = \pi$.

Solution: We start by finding the slope of the tangent line. For this we need the derivative. Apply now quotient rule to see

$$\begin{aligned}f'(x) &= \frac{\cos x e^{\sin x} \cos x + e^{\sin x} \sin x}{\cos^2 x} \\ f'(x) &= \frac{e^{\sin x} \cos^2 x + e^{\sin x} \sin x}{\cos^2 x}\end{aligned}$$

where we have used a chain rule as well. To find the slope of the tangent line at $x = \pi$ we compute

$$f'(\pi) = \frac{e^0(-1)^2 + e^0(0)}{(-1)^2} = 1$$

Now with the slope in hand our equation for the tangent line takes the form (using point-slope form)

$$\begin{aligned}y - y_1 &= m(x - x_1) \\ y - y_1 &= 1(x - x_1).\end{aligned}$$

We have the point $x_1 = \pi$ and so we find

$$y_1 = f(\pi) = \frac{e^0}{-1} = -1$$

All together the equation of the desired tangent line is

$$y + 1 = x - \pi$$

3. For this problem show all relevant limit computations.

3 marks

(a) Matt's favourite function is

$$f(x) = \frac{2}{e^x + e^{-x}}.$$

Find the equation(s) of all horizontal asymptotes of this function.

Solution: To find the H.A. consider the following limits noting that $e^x \rightarrow \infty$ as $x \rightarrow \infty$ and that $e^x \rightarrow 0$ as $x \rightarrow -\infty$

$$\lim_{x \rightarrow \infty} \frac{2}{e^x + e^{-x}} = \frac{2}{\text{"}\infty\text{"} + 0} = 0$$

and

$$\lim_{x \rightarrow -\infty} \frac{2}{e^x + e^{-x}} = \frac{2}{0 + \text{"}\infty\text{"}} = 0.$$

And so we have a H.A. at $y = 0$.

5 marks

(b) Consider now the function

$$g(x) = \frac{\ln x}{x - 2}.$$

Find the equation(s) of all vertical asymptotes of this function. Ensure you determine whether the relevant one sided limits equal $+\infty$ or $-\infty$.

Solution: We first identify the two candidates $x = 2$ and $x = 0$. We suspect $x = 2$ since this is where the denominator is zero. Drawing the graph of $\ln x$ tips us off that there might be an asymptote at $x = 0$. We note that as $x \rightarrow 0^+$, $\ln x \rightarrow -\infty$. We now compute the relevant one sided limits. First around $x = 2$:

$$\lim_{x \rightarrow 2^+} \frac{\ln x}{x - 2} = \frac{\ln(2)}{0^+} = \infty$$

and

$$\lim_{x \rightarrow 2^-} \frac{\ln x}{x - 2} = \frac{\ln(2)}{0^-} = -\infty$$

noting that $\ln(2) > 0$. Now we investigate $x = 0$. We consider only the limit as $x \rightarrow 0^+$ since $g(x)$ is not defined for negative values of x .

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{x - 2} \frac{\text{"}\infty\text{"}}{-2} = \infty.$$

And so we have two V.A. One at $x = 2$ and one at $x = 0$.

- 1 mark 4. (a) Given function $f(x)$ state the definition of its derivative.

Solution: The definition of the derivative is given by the following limit

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

- 4 marks (b) Using the above definition compute the derivative of

$$f(x) = \frac{1}{4x}.$$

No credit will be given for using a method other than the definition of the derivative.

Solution: We compute

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{1}{h} (f(x+h) - f(x)) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{4(x+h)} - \frac{1}{4x} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x}{4(x+h)x} - \frac{(x+h)}{4(x+h)x} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x - x - h}{4(x+h)x} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{4(x+h)x} \right) \\ &= \lim_{h \rightarrow 0} \frac{-1}{4(x+h)x} \\ &= \frac{-1}{4(x+0)x} \\ &= -\frac{1}{4x^2} \end{aligned}$$

which we could confirm with product rule.

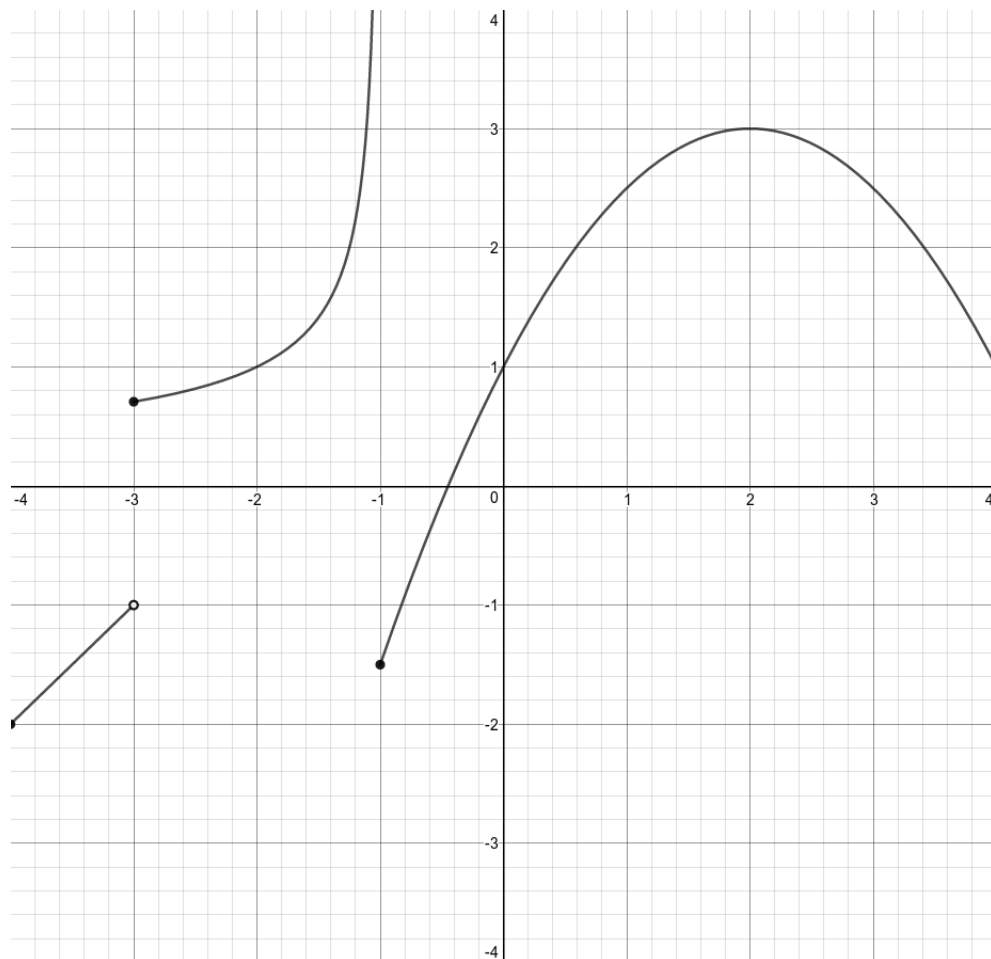
6 marks

5. Sketch the graph of a function satisfying the following properties:

- The domain of $f(x)$ is $\{x \in \mathbb{R} : -4 \leq x \leq 4\}$
- $f(x)$ has a vertical asymptote at $x = -1$
- $\lim_{x \rightarrow -3} f(x)$ does not exist
- $f'(2) = 0$
- $f'(1) > 0$
- $f'(3) < 0$

You do not need to find an equation for your function. Use the axes below.

Solution: There are many graphs that will satisfy the above six conditions. Here is an example of one such graph.



Observe that the slope is positive at $x = 1$, zero at $x = 2$, and negative at $x = 3$. Notice that $f(x)$ is defined for all $x \in [-4, 4]$ including at $x = -1$.