Math 190 Homework 10: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:

1. Compute the following integrals

   (a) $\displaystyle\int \frac{\sin x}{\cos x} dx$

   (b) $\displaystyle\int \sin x e^{\cos x} dx$

   Solution: (a) We make the substitution $u = \cos x$. In this way $du = -\sin x dx$. So

$$\int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du = -\ln|u| + C = -\ln|\cos x| + C.$$

   (b) Again we make the substitution $u = \cos x$. Again $du = -\sin x dx$. So

$$\int \sin x e^{\cos x} dx = -\int e^u du = -e^u + C = -e^{\cos x} + C.$$

   Note in both cases that we can check our answer by taking the derivative (using chain rule).

2. Find a function $F(x)$ such that

$$F'(x) = -\sqrt{3}\cos x + 4\sin 3x$$

   and $F(\pi/3) = 0$.

   Solution:
   We seek the anti-derivative $F(x)$. We compute

$$\begin{aligned} F(x) &= \int \left(-\sqrt{3}\cos x + 4\sin 3x\right) dx \\ &= -\sqrt{3}\int \cos x dx + 4\int \sin 3x dx \\ &= -\sqrt{3}\sin x - \frac{4}{3}\cos 3x + C. \end{aligned}$$

   Now we find C so that our condition is satisfied. We insist that

$$\begin{aligned} 0 = F(\pi/3) &= -\sqrt{3}\sin\left(\frac{\pi}{3}\right) - \frac{4}{3}\cos\left(3\cdot\frac{\pi}{3}\right) + C \\ &= -\sqrt{3}\cdot\frac{\sqrt{3}}{2} - \frac{4}{3}\cos(\pi) + C \\ &= -\frac{3}{2} + \frac{4}{3} + C \end{aligned}$$

and so

$$C = \frac{3}{2} - \frac{4}{3} = \frac{9}{6} - \frac{8}{6} = \frac{1}{6}.$$

Putting everything together we find

$$F(x) = -\sqrt{3}\sin x - \frac{4}{3}\cos 3x + \frac{1}{6}.$$

3. A function is called *odd* if

$$f(-x) = -f(x)$$

for all values of x.

(a) Using a picture, explain why you suspect that

$$\int_{-a}^{a} f(x)dx = 0$$

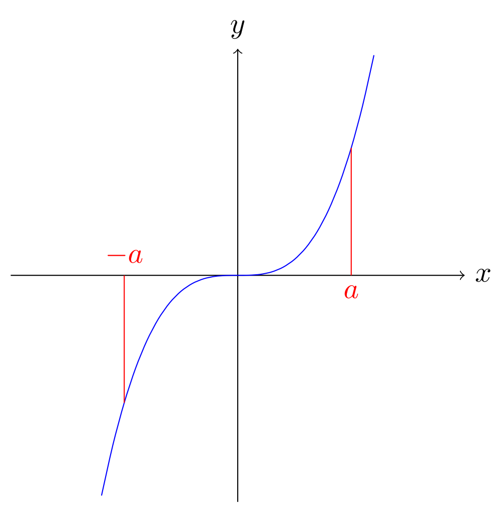
(b) Prove using a substitution that

$$\int_{-a}^{a} f(x)dx = 0$$

for any odd function $f(x)$ and any value of a.

Solution:

(a) Here is the graph of an odd function:



Observe that the magnitude of the area under the curve from $-a$ to 0 is the same as the area under the curve from 0 to a. In this way we expect the negative area to cancel exactly with the positive area. That is

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = 0.$$

(b) We will now prove this fact using a substitution. We first split the integral as follows

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

and then make the substitution $u = -x$ for the first integral only. In this way $du = -dx$ and so (working with just the first integral)

$$\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-u)du.$$

Note that we changed the bounds of integration. That is when $x = -a$ we have $u = a$ and when $x = 0$ we have $u = 0$. Recall that $f(x)$ was defined to be odd. That is to say that $f(-u) = -f(u)$. Substituting this yields

$$\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-u)du = \int_{a}^{0} f(u)du = -\int_{0}^{a} f(u)du$$

where in the last step we have reversed the bounds at the cost of a minus sign. Note also that

$$-\int_{0}^{a} f(u)du = -\int_{0}^{a} f(x)dx$$

since we are just changing the name of our variable. Therefore, after putting everything back together we see

$$\int_{-a}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0$$

as required.

4. Compute the following definite integral

$$\int_{-2}^{2} xe^{x^2} dx.$$

Explain, in reference to Question 3, why you expected this result.

Solution:
We can compute this integral using a substitution. Take $u = x^2$. Then $du = 2xdx$. So

$$\int_{-2}^{2} xe^{x^2} dx = \frac{1}{2}\int_{4}^{4} e^u du = 0$$

since we are starting and ending our integral at the same place, namely $u = 4$. Alternative we can compute the long way:

$$\frac{1}{2}\int_{4}^{4} e^u du = \frac{1}{2}e^u\bigg|_4^4 = \frac{1}{2}\left(e^4 - e^4\right) = 0.$$

If you don't want to change the bounds of integration we can also compute the indefinite integral and substitute the bounds at the end. After the same substitution we get

$$\int xe^{x^2}dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

Note that the $+C$ is not necessary here since we only wish to compute the definite integral. Now

$$\int_{-2}^{2} xe^{x^2}dx = \frac{1}{2}e^{x^2}\bigg|_{-2}^{2} = \frac{1}{2}\left(e^{2^2} - e^{(-2)^2}\right) = \frac{1}{2}\left(e^{2^2} - e^{2^2}\right) = 0.$$

This result is not surprising since the function in question is odd. That fact together with the symmetric limits (-2 to 2) means that the integral must be zero by Question 3.
To compute this integral faster we can prove that it is odd and appeal to Q3. Take

$$f(x) = xe^{x^2}.$$

Observe that

$$f(-x) - xe^{(-x)^2} = -xe^{x^2} = -f(x)$$

and so our function $f(x)$ is odd. In this way we know immediately that

$$\int_{-2}^{2} xe^{x^2}dx = 0.$$

5. Evaluate the following definite integral

$$\int_0^4 \frac{x}{\sqrt{1+2x}}dx.$$

Solution:
We compute by using a substitution. Let $u = 1+2x$. Then $du = 2dx$. Applying the substitution gives

$$\int_0^4 \frac{x}{\sqrt{1+2x}}dx = \frac{1}{2}\int_1^9 \frac{x}{\sqrt{u}}du.$$

We have changed the bounds. Note that when $x = 0$ we have $u = 1$ and when $x = 4$ we have $u = 9$. Our substitution, however, is not complete. We still have an x. We can remove this by using our relationship between u and x:

$$u = 1 + 2x \implies x = \frac{u-1}{2}.$$

And so, our integral now reads

$$\int_0^4 \frac{x}{\sqrt{1+2x}}dx = \frac{1}{2}\int_1^9 \frac{x}{\sqrt{u}}du = \frac{1}{2}\int_1^9 \frac{u-1}{2}\frac{1}{\sqrt{u}}du.$$

We can now compute this integral with a little manipulation

$$\begin{aligned}
\frac{1}{2}\int_1^9 \frac{u-1}{2}\frac{1}{\sqrt{u}}du &= \frac{1}{4}\int_1^9 \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right) du \\
&= \frac{1}{4}\int_1^9 \left(u^{1/2} - u^{-1/2}\right) du \\
&= \frac{1}{4}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\Big|_1^9\right) \quad \text{[alternatively change back to } x \text{ and use old bounds]} \\
&= \frac{1}{2}\left(\frac{1}{3}u^{3/2} - u^{1/2}\Big|_1^9\right) \\
&= \frac{1}{2}\left(\frac{1}{3}(\sqrt{9})^3 - \sqrt{9} - \frac{1}{3}1^{3/2} + 1^{1/2}\right) \quad \text{[feel free to stop here]} \\
&= \frac{1}{2}\left(\frac{1}{3}27 - 3 - \frac{1}{3} + 1\right) \\
&= \frac{1}{2}\left(9 - 2 - \frac{1}{3}\right) \\
&= \frac{1}{2}\left(7 - \frac{1}{3}\right) \\
&= \frac{1}{2}\cdot\frac{20}{3} \\
&= \frac{10}{3}.
\end{aligned}$$