HW 11 : Solutions.

1. • $\int x e^x \, dx$ let $u = x$, $dv = e^x dx$; $\frac{du}{dx} = 1$, $\int dv = \int e^x dx$; $du = dx$, $v = e^x$

$= xe^x - \int e^x \, dx$

$= xe^x - e^x + C.$

• $\underbrace{\int x \sin x \, dx}_{(*)}$ let $u = x$, $dv = \sin x \, dx$; $du = dx$, $v = -\cos x$

$= -x\cos x + \int \cos x \, dx$

$= -x \cos x + \sin x + C.$

• $\int x^2 \cos x \, dx$ let $u = x^2$, $dv = \cos x \, dx$; $du = 2x\,dx$, $v = \sin x$

$= x^2 \sin x - 2\underbrace{\int x \sin x \, dx}_{(*)}$

$= x^2 \sin x - 2\left(-x\cos x + \sin x\right) + C.$

• $\int (x+3) e^{-x} \, dx$ let $u = x+3$, $dv = e^{-x} dx$; $du = dx$, $v = -e^{-x}$

$= -(x+3)e^{-x} + \int e^{-x} \, dx$

$= -(x+3)e^{-x} - e^{-x} + C.$

• $\int x \sin(2x) \, dx$ let $u = x$, $dv = \sin(2x) dx$; $du = dx$, $v = \frac{-\cos(2x)}{2}.$

$= -\frac{x}{2} \cos(2x) + \frac{1}{2}\int \cos(2x) \, dx$

$= -\frac{x}{2} \cos(2x) + \frac{1}{4} \sin(2x) + C.$

(2)

- $\int x\cos(3x+\pi)\,dx$ let $u = x$, $dv = \cos(3x+\pi)dx$; $du = dx$, $v = \frac{\sin(3x+\pi)}{3}$

$= \frac{x}{3}\sin(3x+\pi) - \frac{1}{3}\int \sin(3x+\pi)\,dx$

$= \frac{x}{3}\sin(3x+\pi) + \frac{1}{9}\cos(3x+\pi) + C.$

- $\int x\ln x\,dx$ let $u = \ln x$, $dv = x\,dx$; $du = \frac{1}{x}dx$, $v = \frac{x^2}{2}$.

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x^2 \frac{1}{x}\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{1}{2}\frac{x^2}{2} + C.$

- $\int \frac{\ln x}{x^2}\,dx$ let $u = \ln x$, $dv = \frac{1}{x^2}dx$; $du = \frac{1}{x}dx$, $v = -\frac{1}{x}$.

$= -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx$

$= -\frac{1}{x}\ln x - \frac{1}{x} + C.$

- $\int x\ln(1+x)\,dx$ let $u = \ln(1+x)$, $dv = x\,dx$; $du = \frac{1}{1+x}dx$, $v = \frac{x^2}{2}$.

$= \frac{x^2}{2}\ln(1+x) - \frac{1}{2}\int \frac{x^2}{1+x}\,dx$

Now let $u = 1+x \Rightarrow x^2 = (u-1)^2$.
$du = dx$

③

$$= \frac{x^2}{2}\ln(1+x) - \frac{1}{2}\int \frac{(u-1)^2}{u}\,du$$

$$= \frac{x^2}{2}\ln(1+x) - \frac{1}{2}\int \frac{u^2-2u+1}{u}\,du$$

$$= \frac{x^2}{2}\ln(1+x) - \frac{1}{2}\int \left(u - 2 + \frac{1}{u}\right) du$$

$$= \frac{x^2}{2}\ln(1+x) - \frac{1}{2}\left[\frac{u^2}{2} - 2u + \ln u\right] + C$$

$$= \frac{x^2}{2}\ln(1+x) - \frac{1}{2}\left(\frac{(1+x)^2}{2} - 2(1+x) + \ln(1+x)\right) + C.$$

- $\overbrace{\int \ln x\,dx}^{(*)}$ let $u = \ln x$, $du = \frac{1}{x}dx$, $dv = dx$, $v = x$.

$$= x\ln x - \int \frac{x}{x}\,dx$$

$$= x\ln x - \int dx = x\ln x - x + C.$$

- $\int (\ln x)^2\,dx$ let $u = \ln x$, $du = \frac{1}{x}dx$, $\cancel{dv = dx}$, $\cancel{v = x}$

$dv = \ln x\,dx$
$v = \int \ln x\,dx$ $\{(*)$
$= x\ln x - x.$

$$\cancel{= x\ln x - \int (x\ln x - x)\frac{1}{x}\,dx}$$

$$\cancel{= x\ln x}$$

$$= \ln x\,(x\ln x - x) - \int (x\ln x - x)\frac{1}{x}\,dx$$

$$= x(\ln x)^2 - x\ln x - \underbrace{\int \ln x\,dx}_{(*)} + \int dx$$

$$= x(\ln x)^2 - x\ln x - (x\ln x - x) + x + C.$$

$$= x(\ln x)^2 - 2x\ln x + 2x + C \quad //$$

(4)

- $\int e^x \cos x \, dx$ — let $u = e^x$, $du = e^x dx$, $dv = \cos x \, dx$, $v = \sin x$

$= e^x \sin x - \int e^x \sin x \, dx$

let $u = e^x$, $dv = \sin x \, dx$, $du = e^x dx$, $v = -\cos x$

$= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx\right)$

$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$ what we started with.

$\Rightarrow \int e^x \cos x \, dx + \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$

$2\int e^x \cos x \, dx = e^x(\sin x + \cos x)$

$\int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x) + C.$

- $\int \cos x \ln(\sin x) \, dx$ — let $u = \ln(\sin x)$, $du = \frac{1}{\sin x}\cos x \, dx$, $dv = \cos x \, dx$, $v = \sin x$

$= \sin x \ln(\sin x) - \int \frac{\sin x}{\sin x} \cos x \, dx$

$= \sin x \ln(\sin x) - \int \cos x \, dx$

$= \sin x \ln(\sin x) - \sin x + C.$

OR/ by Substitution. Let $u = \sin x$, $du = \cos x \, dx$

- $\int \cos x \ln(\sin x) dx = \int \ln(u) du$ → taken from last page.

$= u\ln(u) - u + C = \sin x \ln(\sin x) - \sin x + C$

5

2. $\bullet \displaystyle\int_{\ln 2}^{\ln 3} x e^{x}\, dx = xe^{x} - e^{x} \Big|_{\ln 2}^{\ln 3}$

$$= \ln 3\, e^{\ln 3} - e^{\ln 3} - \left(\ln 2\, e^{\ln 2} - e^{\ln 2}\right)$$

$$= 3\ln 3 - 3 - 2\ln 2 + 2$$

$$= 3\ln 3 - 2\ln 2 - 1.$$

$\bullet \displaystyle\int_{0}^{\pi/2} x\sin x\, dx = -x\cos x + \sin x \Big|_{x=0}^{\pi/2}$

$$= -\frac{\pi}{2}\cos(\pi/2) + \sin(\pi/2) + 0\cdot\cos(0) - \sin(0)$$

$$= 0 + 1 + 0 - 0 = 1.$$

$\bullet \displaystyle\int_{1}^{2} \frac{\ln x}{x^2}\, dx = -\frac{1}{x}\ln x - \frac{1}{x} \Big|_{x=1}^{2}$

$$= -\frac{1}{2}\ln 2 - \frac{1}{2} + \frac{1}{1}\ln(1) + \frac{1}{1}$$

$$= -\frac{1}{2}\ln 2 - 1/2 + 1$$

$$= -\frac{1}{2}\ln 2 + 1/2.$$

6

rate of change of volume of oil.

3. $r(t) = 100e^{-0.01t}$.

volume of oil $\rightarrow V(t) = \int r(t)dt$

$= 100\int e^{-0.01t}dt$

$= \frac{100}{-0.01}e^{-0.01t} + C$.

$= -\frac{100}{0.01}e^{-0.01t} + C$

Oil starts leaking at $t=0$ so $V(0) = 0$.

$V(0) = 0 = \frac{-100}{0.01}e^{0} + C$

$\Rightarrow C = \frac{+100}{0.01}$.

$\Rightarrow V(t) = \frac{-100}{0.01}e^{-0.01t} + \frac{100}{0.01}$.

After 60 minutes:

$V(60) = \frac{-100}{0.01}e^{-0.01 \cdot 60} + \frac{100}{0.01}$.

$\simeq 4511$ L.

7

4. $\bar{v}(t) = 2\cos(\pi t) + t.$

$s(t) = \int v(t)$

$= 2\int \cos(\pi t)\,dt + \int t\,dt$

$= \frac{2}{\pi}\sin(\pi t) + \frac{t^2}{2} + C.$

We know $s(0) = 10$

$10 = \frac{2}{\pi}\sin(0) + \frac{0^2}{2} + C$

$\Rightarrow \quad C = 10.$

$s(t) = \frac{2}{\pi}\sin(\pi t) + \frac{t^2}{2} + 10.$

Want: $s(1.5) = \frac{2}{\pi}\sin\left(\frac{3\pi}{2}\right) + \frac{\left(\frac{3}{2}\right)^2}{2} + 10$

$= \frac{-2}{\pi} + \frac{9}{8} + 10$

$\simeq 10.48$ cm.

8

5. $a(t) = -g$

$$v(t) = \int a(t)\,dt$$
$$= -\int g\,dt = -gt + C.$$

$$v(0) = 2. = -g \cdot 0 + C \Rightarrow C = 2$$

$$v(t) = -gt + 2.$$

Find: $v(t) = 0 = -gt + 2$

$$\Rightarrow t = 2/g \; . \simeq 0.2 \text{ seconds}$$
(if $g \simeq 10$).

6. $a(t) = -50\,000$
$$v(t) = 50\,000t + C.$$
Know: $v(0) = 100 \Rightarrow C = 100.$
$$v(t) = 50\,000t + 100$$
Find: $v(t) = 0 = 50\,000t + 100$
$$t = \frac{100}{50\,000} = \frac{1}{500} \text{ hours.}$$
$$\simeq 7.2 \text{ Seconds.}$$

Do you get ice cream?
$$s(t) = \int v(t)\,dt$$
$$= \int(-50\,000t + 100)\,dt$$
$$= -\frac{50\,000}{2}t^2 + 100t + C.$$

At $t=0$ you have travelled 0km so
$$s(0) = 0 = -\frac{50\,000}{2}(0)^2 + 100(0) + C \Rightarrow C = 0.$$

9

6. (Cont.)

So, $S(t) = \frac{-50\,000}{2} t^2 + 100t$.

After $\frac{1}{500}$ hours you have gone:

$$S\left(\frac{1}{500}\right) = \frac{-50\,000}{2}\left(\frac{1}{500}\right)^2 + \frac{100}{500}$$

$$= -\frac{50\,000}{2} \cdot \frac{1}{250\,000} + 1/5.$$

$$= -\frac{1}{25} + 1/5 = -\frac{1}{10} + \frac{2}{10}$$

$$= 1/10 \text{ km}$$

$$= 100\text{m}$$

$\Rightarrow$ You stop exactly in time for ice cream.