Math 190 Homework 1: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:

1. Find the domain of

$$f(x) = \frac{\sqrt{x+7}}{x^2+3x-18}.$$

Solution: We require $x + 7 \geq 0$ and $x^2 + 3x - 18 \neq 0$. The first so that we are not taking the square root of a negative number and the second so that we are not dividing by zero. Rearranging the first gives $x \geq -7$. For the second we factor

$$\begin{aligned} x^2 + 3x - 18 &\neq 0 \\ (x+6)(x-3) &\neq 0 \end{aligned}$$

and so $x \neq -6$ and $x \neq 3$. Putting everything together we see that the domain is

$$\{x \in \mathbb{R} : x \geq -7 \text{ and } x \neq -6 \text{ and } x \neq 3\}.$$

This can also be denoted as

$$[-7, -6) \cup (-6, 3) \cup (3, \infty).$$

2. Find all (real) zeros of

$$g(x) = \begin{cases} 2x^2 - 7x + 3, & x \leq 2 \\ -\frac{1}{2}(x-2) + 2, & x > 2 \end{cases}.$$

Solution: Let us find where each branch is zero. First

$$\begin{aligned} 2x^2 - 7x + 3 &= 0 \\ 2x^2 - 6x - x + 3 &= 0 \\ 2x(x-3) - (x-3) &= 0 \\ (2x-1)(x-3) &= 0 \end{aligned}$$

and so we see $x = 3$ and $x = 1/2$. Since $g(x)$ only takes the value $2x^2 - 7x + 3$ for $x \leq 2$ our computation does *not* imply that $g(3) = 0$. We have, however, found that $g(1/2) = 0$. Now for the second branch we solve

$$\begin{aligned} -\frac{1}{2}(x-2) + 2 &= 0 \\ x - 2 &= 4 \\ x &= 6. \end{aligned}$$

Since $6 > 2$ we do in fact have $g(6) = 0$. Therefore our two zeros are:

$$x = \frac{1}{2} \text{ and } x = 6.$$

3. Find all (real) values of θ satisfying

$$3\cos(4\theta) - \pi = 0$$

where $0 \leq \theta < 2\pi$.

Solution 1: First let us rearrange the equation to read

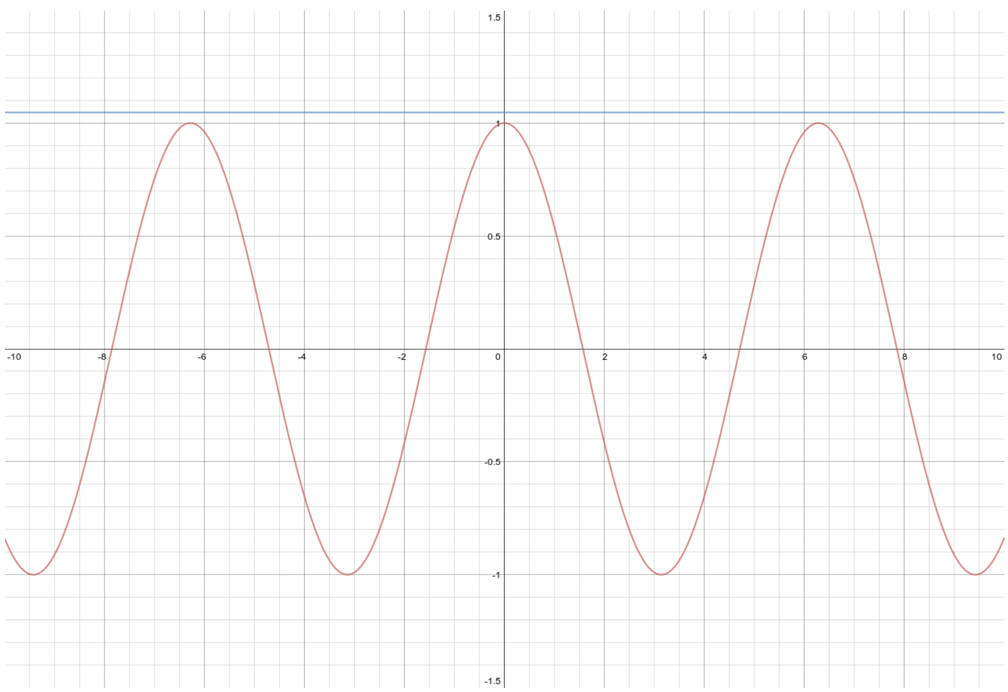
$$\cos(4\theta) = \frac{\pi}{3}.$$

We can even let $x = 4\theta$ if we wish to see

$$\cos x = \frac{\pi}{3}.$$

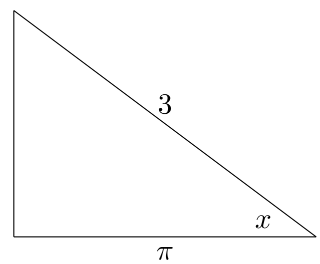
Recall the range of $\cos x$ and that $-1 \leq \cos x \leq 1$. Note also that $\pi = 3.14\ldots$ and so $\pi/3 > 1$. In this way we see that no (real) value for x will satisfy $\cos x = \pi/3$. Hence our original equation has no solutions.

Solution 2: Starting from $\cos x = \pi/3$ we can proceed graphically. Let us plot the graphs $y = \cos x$ and $y = \pi/3$ to see where (if at all) they intersect.



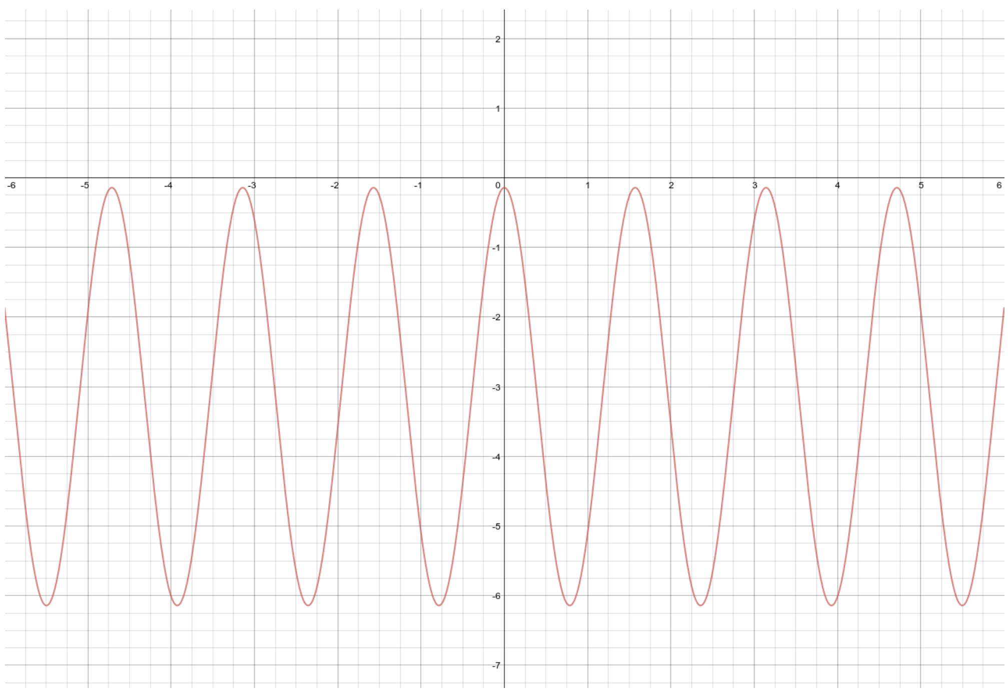
So as seen in the graph there is no intersection between $y = \pi/3$ (blue) and $y = \cos x$ (red) and therefore our original equation has no (real) solutions.

Solution 3: Again starting from $\cos x = \pi/3$ we can solve this problem by considering a right triangle with angle x, hypotenuse 3 and adjacent side length π:



However such a triangle cannot exist since the hypotenuse must always be greater than the lengths of the other two sides. In this way we again see that no angle x will result in $\cos x$ taking the value $\pi/3$.

Solution 4: We can even plot $f(x) = 3\cos(4\theta) - \pi$ from the start using transformations to see:



From the graph see that $f(x)$ does not take the value 0 for any x value and so the equation has no solution.

4. Find all (real) solutions to

$$x^4 - 4x^2 + 2 = 0.$$

Solution: While solving a general quartic equation is a difficult task this particular one is special. Observe that if we let $u = x^2$ then we obtain the quadratic equation

$$y^2 - 4y + 2 = 0$$

which we can solve in the usual way. We proceed by means of the quadratic formula

$$\begin{aligned} y &= \frac{4 \pm \sqrt{4^2 - 4(1)(2)}}{2} \\ &= 2 \pm \frac{\sqrt{8}}{2} \\ &= 2 \pm \sqrt{2}. \end{aligned}$$

Now that we have solved for y we can find x by taking the square roots. Note that since $2 - \sqrt{2} > 0$ we don't have to worry about taking the square root of a negative number. So our four solutions are:

$$x = \pm\sqrt{y} = \pm\sqrt{2 \pm \sqrt{2}}.$$

5. Explain why the equation

$$x^3 + ax^2 + bx + c = 0$$

cannot have four solutions no matter the values of a, b, c.

Solution: Let us start by supposing that $f(x) = x^3 + ax^2 + bx + c$ has exactly four zeros. Name them x_1, x_2, x_3, x_4. If this were the case, then after some factoring, we should be able to write $f(x)$ as

$$f(x) = A(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

where A is a number. Now if we expand the above expression the first term will contain x^4. That is

$$f(x) = Ax^4 + \dots.$$

But $f(x)$ cannot take this form since it is a cubic equation. Therefore our initial assumption was false and f cannot have four zeros, as desired.