Math 190 Homework 2: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner on stapled pieces of paper. Your stapled work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page. Staple your assignment.

Questions:

1. Find all $x \in [0, 2\pi)$ satisfying

$$\sin x \cos x = \sqrt{2} \sin x.$$

Solution: We can cancel $\sin x$ from both sides of the equation providing $\sin x \neq 0$ (or else we would be dividing by zero). In this way we must consider the equation $\cos x = \sqrt{2}$ as well as the possibility that $\sin x = 0$. Alternatively we can rearrange the equation to read

$$\sin x \cos x - \sqrt{2} \sin x = 0$$

and factor to see

$$\sin x(\cos x - \sqrt{2}) = 0.$$

Either way we must solve the two equations

$$\cos x = \sqrt{2}$$
$$\sin x = 0.$$

For the first, if we recall the range of $\cos x$ we see that $-1 \leq \cos x \leq 1$ and note that $\sqrt{2} > 1$ we observe that $\cos x = \sqrt{2}$ has no real solutions. We may still have solutions from $\sin x = 0$, however. Recall (perhaps after plotting the graph of sine) that $\sin x = 0$ for $x = n\pi$ where n an integer (also denoted $n \in \mathbb{Z}$ or $n = \ldots - 2, -1, 0, 1, 2, \ldots$). Therefore all the solutions to our original equation are $x = n\pi$ with $n \in \mathbb{Z}$.

2. Find all $x \in [0, 2\pi)$ satisfying

$$2\cos^2 x + (2 + \sqrt{3}) \cos x = -\sqrt{3}.$$

Solution: First, let us rearrange the equation to read

$$2\cos^2 x + (2 + \sqrt{3}) \cos x + \sqrt{3} = 0$$

and if we let $y = \cos x$ we recognize the above as a quadratic equation

$$2y^2 + (2 + \sqrt{3})y + \sqrt{3} = 0.$$

Rather than use the quadratic formula (which can be done) we will factor the above by decomposition. First we split the middle term to see

$$2y^2 + 2y + \sqrt{3}y + \sqrt{3} = 0$$

and factor

$$2y(y+1) + \sqrt{3}(y+1) = 0$$
$$(2y+\sqrt{3})(y+1) = 0.$$

Now we can put back $\cos x$ and see

$$(2\cos x + \sqrt{3})(\cos x + 1) = 0.$$

Observing the graph of cosine we see that for $x \in [0, 2\pi)$ the only way to have $\cos x = -1$ is if $x = \pi$. Similarly we solve $\cos x = -\sqrt{3}/2$ which gives us $x = 5\pi/6$ and $x = 7\pi/6$ after observing the unit circle and/or special triangles. So all together we have three solutions

$$x = \frac{5\pi}{6}, \pi, \frac{7\pi}{6}.$$

3. Find all (real) zeros of the function

$$h(x) = \sin\left(\frac{1}{x}\right).$$

Solution: First note that the domain of h is $\{x \in \mathbb{R} : x \neq 0\}$. To simplify matters we can let $\theta = 1/x$. In this way we must solve

$$\sin\theta = 0.$$

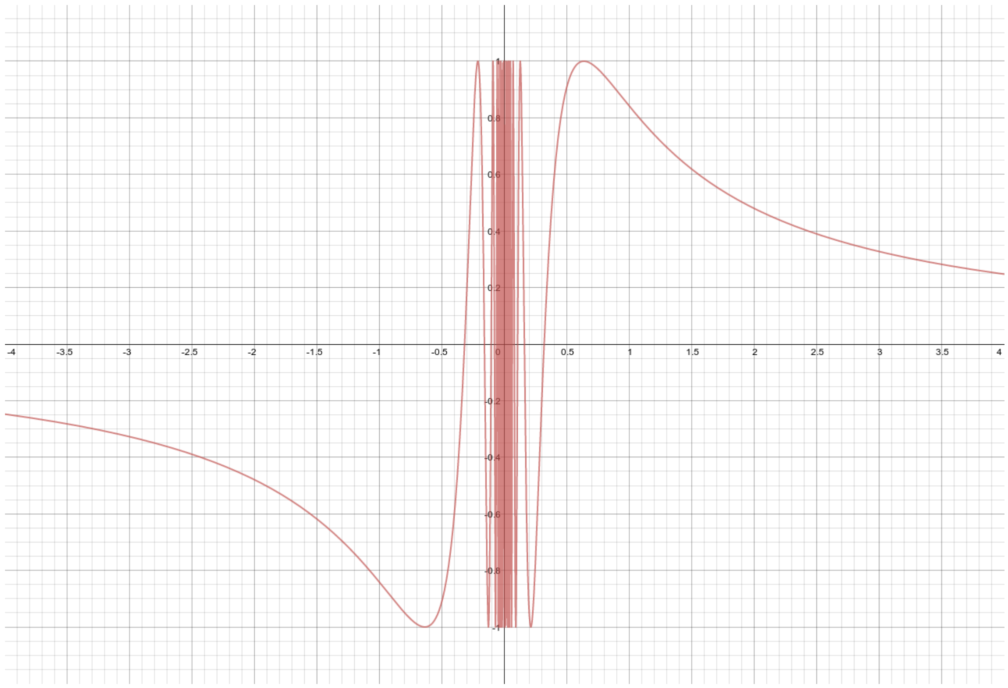
We have already seen that such θ take the form $\theta = n\pi$ where n is an integer. In this way we see that

$$\frac{1}{x} = \theta = n\pi$$

or rather

$$x = \frac{1}{n\pi}.$$

Now we cannot have n be all the integers since if $n = 0$ then x is not defined. So we have the zeros $x = 1/(n\pi)$ where n is a non-zero integer (also written as $n \in \mathbb{Z} \setminus \{0\}$ or $n = \ldots -2, -1, 1, 2, \ldots$). By the way, here is the graph of $\sin 1/x$. Note the largest positive zero at $x = 1/\pi$ followed by $1/(2\pi)$ and $1/(3\pi)$ and so on.



4. Final all (real) x satisfying

$$2x^{1/3} + 5x^{4/3} = 0.$$

Solution: Let us common factor. Each term has an $x^{1/3}$. So

$$x^{1/3}(2 + 5x) = 0$$

noting that $4/3 - 1/3 = 3/3 = 1$. So we solve both $x^{1/3} = 0$ and $2 + 5x = 0$. For the first if we cube both sides we achieve $x = 0$. For the second we rearrange to obtain $x = -2/5$. And so our two solutions are

$$x = -\frac{2}{5}, 0.$$

5. Let

$$f(x) = \sqrt{2x+1} \quad \text{and} \quad g(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0 \end{cases}$$

(a) The composition $f(g(x))$ has only two possible outputs. Find the two values and explain your answer.

(b) The composition $g(f(x))$ also only has two outputs. Find these values as well. Explain.

Solution: (a) No matter the input x the value $g(x)$ will take one of only three values. If $x < 0$ then $g(x) = -1$ and so

$$f(g(x)) = \sqrt{2(-1) + 1} = \sqrt{-1}$$

is not defined (for us). If $x = 0$ then $g(x) = 0$ and so

$$f(g(x)) = \sqrt{2(0)+1} = \sqrt{1} = 1$$

which is a fine number. Finally if $x > 0$ then $g(x) = 1$ and so

$$f(g(x)) = \sqrt{2(1)+1} = \sqrt{3}$$

another fine number. So we see that $f(g(x))$ will take only two values no matter the input x. These two values are 1 and $\sqrt{3}$. By the way (this is not required) after this discussion we can write the following equation for $f(g(x))$

$$f(g(x)) = \begin{cases} \text{undefined} & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x = 0 \\ \sqrt{3} & \text{if} \quad x > 0. \end{cases}$$

(b) We play the same game. Let us think about the outputs from the function $f(x)$. Noting the range of $\sqrt{x}$ we know $0 \leq \sqrt{x}$. So $f(x) = \sqrt{2x+1}$ can only take positive values or zero. If $f(x) = 0$ then $g(f(x)) = 0$. If $f(x) > 0$ then $g(f(x)) = 1$. And so the two possible outputs are 0 and 1. With a little extra work (again not required) we can find an equation for $g(f(x))$. To have $0 = f(x) = \sqrt{2x+1}$ we would need $2x + 1 = 0$ that is $x = -1/2$. If $x < -1/2$ then $2x + 1 < 0$ and so $f(x)$ is not defined. If $x > -1/2$ then $f(x) > 0$ and so $g(f(x)) = 1$. Putting everything together gives

$$g(f(x)) = \begin{cases} \text{undefined} & \text{if} \quad x < -\frac{1}{2} \\ 0 & \text{if} \quad x = -\frac{1}{2} \\ 1 & \text{if} \quad x > -\frac{1}{2}. \end{cases}$$