Math 190 Homework 3: Due Monday October 5

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:

1. Find all x satisfying

$$\ln(2x-3)-\ln(5)=8.$$

Solution: We first rewrite the above equation using one of our logarithm laws:

$$\begin{aligned} \ln(2x-3)-\ln(5)&=8 \\ \ln\left(\frac{2x-3}{5}\right)&=8. \end{aligned}$$

Note that we **cannot** take e to the power of each *term*. We must take e to the power of each *side*. Continuing we see

$$\begin{aligned} e^{\ln\left(\frac{2x-3}{5}\right)}&=e^8 \\ \frac{2x-3}{5}&=e^8 \\ 2x-3&=5e^8 \\ 2x&=5e^8+3 \\ x&=\frac{5e^8+3}{2}. \end{aligned}$$

And so, we have arrived at the desired answer.

2. Find all x satisfying

$$\ln\left((x-1)^x\right)=0.$$

Solution: Again, let us start by using a logarithm identity

$$\begin{aligned} \ln\left((x-1)^x\right)&=0 \\ x\ln(x-1)&=0. \end{aligned}$$

At this stage we can divide both sides by x providing that $x\neq 0$. If $x=0$ we can observe that our original equation is actually satisfied. That is

$$\ln((0-1)^0)=\ln(1)=0$$

and so $x = 0$ is a solution. Also acceptable is to say that at the stage $x \ln(x-1) = 0$ we have two solutions, either $x = 0$ or $\ln(x-1) = 0$. (If plugging $x = 0$ into our equation to see $0 \cdot \ln(0-1)$ makes you nervous then I congratulate you on noticing that we can't put -1 into ln. Everything does work out though, so feel free to ask me about it.)

So, we have the solution $x = 0$. If $x \neq 0$ then we can divide by x to see

$$\ln(x-1) = 0$$

and notice that the only way to make the logarithm zero is to take $x - 1 = 1$ and so $x = 2$ is our other solution. Together we have two solutions: $x = 0$ and $x = 2$.

3. Find all x satisfying

$$e^{2x} - 4e^x + 4 = 0.$$

Solution:

This problem may look intimidating from the onset but if we looks closely we can recognize the above as a familiar quadratic equation. First, let us rewrite the first term in our expression using exponent rules

$$\begin{aligned} e^{2x} - 4e^x + 4 &= 0 \\ (e^x)^2 - 4e^x + 4 &= 0. \end{aligned}$$

Now if we let $u = e^x$ we see

$$u^2 - 4u + 4 = 0$$

which we can factor as

$$(u-2)^2 = 0.$$

So, we obtain the solution $u = 2$. Switching back to x we get

$$\begin{aligned} e^x &= 2 \\ \ln(e^x) &= \ln 2 \\ x &= \ln 2. \end{aligned}$$

Hence our only solution is $x = \ln 2$.

4. Many natural phenomena obey power rules. That is

$$Y = CX^m$$

where C and m are constants which depend on the particular application. For example in physics we have the Stephan-Boltzmann equation where Y is the power emitted by a star with temperature X. In forestry we have models of tree size distribution where Y is the number of trees with stem size X. Other examples include frequency of words in most languages, population of cities, and rate of reaction in chemistry.

(a) Let $y = \ln Y$ and $x = \ln X$. Express y in terms of x assuming that $Y = CX^m$. Note that C and m are fixed constants.

(b) Suppose we made a plot of y as a function of x. What would the graph look like?

Solution: (a) Let us start with $Y = CX^m$. Since we want to introduce x and y we take the natural logarithm of both sides

$$\ln Y = \ln\left(CX^m\right).$$

The left side is exactly y. For the right side we apply log rules:

$$\begin{aligned} y &= \ln C + \ln X^m \\ &= \ln C + m \ln X. \end{aligned}$$

At this stage we have recognized $x = \ln X$ in our equation and so we make the substitution

$$y = \ln C + mx.$$

We have now achieved y as a function of x as desired.

(b) Since C is a fixed constant the number $\ln C$ is also a fixed constant. We can recognize our equation for y as taking the form $y = mx + b$ where m is m and b is $\ln C$. In this way we see that our equation is the equation of a line with y-intercept $\ln C$ and slope m.

5. In this problem you will prove the identity

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

as seen in class. First let $z_1 = \log_b(x)$ and $z_2 = \log_b(y)$. Rewrite these two equations using exponents instead of logarithms. Use your knowledge of exponent rules to manipulate the equations until you achieve $z_1 + z_2 = \log_b(xy)$. Make sure that you explain each step.

Solution: There are several ways to formulate the proof. Here is one way. Let $z_1 = \log_b(x)$ and $z_2 = \log_b(y)$. That is to say that $b^{z_1} = x$ and $b^{z_2} = y$. Let us now consider the product

$$xy = b^{z_1} b^{z_2}$$

and use exponent rules to see

$$xy = b^{z_1 + z_2}.$$

Switching this exponential equation back to a logarithm equation gives

$$\log_b(xy) = z_1 + z_2.$$

Recalling the definition of z_1 and z_2 we arrive at the desired identity

$$\log_b(xy) = \log_b(x) + \log_b(y).$$

6. **Bonus** Prove the other two logarithm identities.

Solution: First we show $\log_b(x/y) = \log_b(x) - \log_b(y)$. This proof proceeds in the same manner as the previous. Let $z_1 = \log_b(x)$ and $z_2 = \log_b(y)$ that is to say $b^{z_1} = x$ and $b^{z_2} = y$. Consider now the quotient and use the exponent rule to see

$$\frac{x}{y} = \frac{b^{z_1}}{b^{z_2}} = b^{z_1 - z_2}.$$

Switching back to a logarithm equation we have

$$\log_b\left(\frac{x}{y}\right) = z_1 - z_2 = \log_b(x) - \log_b(y).$$

With the required result in hand the proof is complete.

Now the third identity: $\log_b(x^p) = p\log_b x$. To start let $z = p\log_b(x)$. Let us manipulate this equation using our knowledge of logarithms and exponent laws:

$$\begin{aligned}
\frac{z}{p} &= \log_b(x) \\
b^{z/p} &= x \\
(b^z)^{1/p} &= x \\
b^z &= x^p \\
z &= \log_b(x^p)
\end{aligned}$$

We have therefore achieved

$$p\log_b(x) = \log_b(x^p)$$

as required.