Math 190 Homework 4: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:
When asked to compute a limit in the following problems: Find the value of the limit if it exists. If the limit does not exist but you can assign the value ∞ or $-\infty$ to the limit do so. Otherwise, explain why the limit does not exist.

1. Compute
$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right).$$

   Solution: We first find a common denominator and then multiply by the conjugate. Observe
$$\begin{aligned}
\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right) &= \lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{\sqrt{1+t}}{t\sqrt{1+t}}\right)\\
&= \lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}\cdot\frac{1+\sqrt{1+t}}{1+\sqrt{1+t}}\\
&= \lim_{t\to 0}\frac{1-(1+t)}{t\sqrt{1+t}(1+\sqrt{1+t})}\\
&= \lim_{t\to 0}\frac{-t}{t\sqrt{1+t}(1+\sqrt{1+t})}\\
&= \lim_{t\to 0}\frac{-1}{\sqrt{1+t}(1+\sqrt{1+t})}\\
&= \frac{-1}{\sqrt{1+0}(1+\sqrt{1+0})}\\
&= -\frac{1}{2}.
\end{aligned}$$
   Once the t's cancelled we had no problem to substitute $t=0$. Hence, the limit exists and takes value $-1/2$.

2. Compute
$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{|x|}\right).$$

   Solution: We consider the two one sided limits on account of the absolute value. Recall that
$$|x| = \begin{cases} x & \text{if } x \geq 0\\ -x & \text{if } x < 0\end{cases}$$

and consider first the limit from the right

$$\begin{aligned}\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right) &= \lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x}\right)\\ &= \lim_{x\to 0^+}\left(\frac{0}{x}\right)\\ &= \lim_{x\to 0^+} 0\\ &= 0.\end{aligned}$$

Note that we have replaced $|x|$ with x since $x > 0$. Now for $x < 0$ we replace $|x|$ with $-x$

$$\begin{aligned}\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right) &= \lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{-x}\right)\\ &= \lim_{x\to 0^-}\left(\frac{1}{x}+\frac{1}{x}\right)\\ &= \lim_{x\to 0^-}\frac{2}{x}\\ &= -\infty.\end{aligned}$$

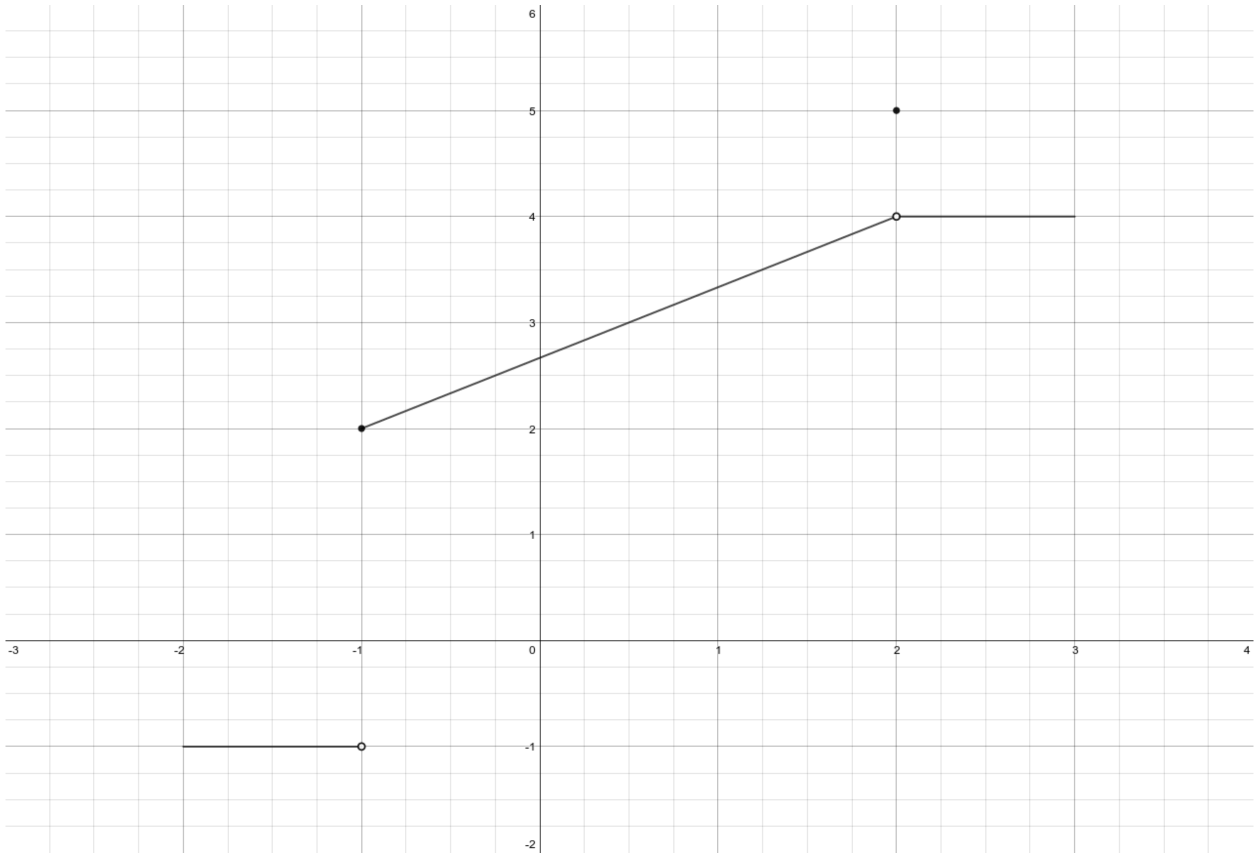
And so, we conclude that the full limit does not exist (nor does it 'equal' ∞ or $-\infty$) since the one sided limits are not equal.

3. Draw the graph of a function $f(x)$ satisfying the following properties (you do not have to come up with an equation for your graph).
   - The domain is $\{x \in \mathbb{R} : -2 \leq x \leq 3\}$.
   - $\lim_{x\to 2} f(x) = 4$
   - $f(2) = 5$
   - $\lim_{x\to -1^+} f(x) = 2$
   - $\lim_{x\to -1^-} f(x) = -1$

   Solution: There are many functions that satisfy the above properties. Here is one example (see below). By the way, here is an equation for the below graph

$$f(x) = \begin{cases} 2, & -2 \leq x < -1 \\ \frac{2}{3}x + \frac{8}{3}, & -1 \leq x < 2 \\ 5, & x = 4 \\ 4, & 2 < x \leq 3 \end{cases}$$

Note that it satisfies the vertical line test and is in fact a function.



4. Find the equations of all vertical and horizontal asymptotes of the following function

$$f(x) = \frac{3x^2 - 14x - 5}{2x^2 - 9x - 5}.$$

Ensure you show the computation of all relevant limits.

Solution

Let us start by finding the horizontal asymptotes. To do so we inspect the limits as $x \to \pm\infty$. First we divide both the top and bottom by the highest power

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^2 - 14x - 5}{2x^2 - 9x - 5} &= \lim_{x\to\infty} \frac{3x^2/x^2 - 14x/x^2 - 5/x^2}{2x^2/x^2 - 9x/x^2 - 5/x^2} \\
&= \lim_{x\to\infty} \frac{3 - 14/x - 5/x^2}{2 - 9/x - 5/x^2} \\
&= \frac{3 - 0 - 0}{2 - 0 - 0} \\
&= \frac{3}{2}.
\end{aligned}$$

So, $f(x)$ approaches $3/2$ as x approaches infinity. We therefore have a horizontal asymptote with equation $y = 3/2$. Computing the limit as x approaches negative infinity is almost identical

$$\begin{aligned}\lim_{x\to-\infty}\frac{3x^2-14x-5}{2x^2-9x-5} &= \lim_{x\to-\infty}\frac{3x^2/x^2-14x/x^2-5/x^2}{2x^2/x^2-9x/x^2-5/x^2}\\ &= \lim_{x\to-\infty}\frac{3-14/x-5/x^2}{2-9/x-5/x^2}\\ &= \frac{3-0-0}{2-0-0}\\ &= \frac{3}{2}.\end{aligned}$$

With this computation in hand we see that $f(x)$ has only one horizontal asymptote with equation $y = 3/2$.

Now we turn to the vertical asymptotes. We must first identify candidates for vertical asymptotes. We do this by finding the zeros of the denominator. The zeros become apparent after factoring

$$f(x) = \frac{3x^2-14x-5}{2x^2-9x-5} = \frac{3x^2-14x-5}{(2x+1)(x-5)}.$$

So, our candidates for V.A. are $x = 5$ and $x = -1/2$. Let us first deal with $x = 5$. To figure out if we have an asymptote or not we take the limits as x approaches 5 from both sides

$$\begin{aligned}\lim_{x\to5^-}\frac{3x^2-14x-5}{2x^2-9x-5} &= \lim_{x\to5^-}\frac{(3x+1)(x-5)}{(2x+1)(x-5)}\\ &= \lim_{x\to5^-}\frac{3x+1}{2x+1}\\ &= \frac{3(5)+1}{2(5)+1}\\ &= \frac{16}{11}.\end{aligned}$$

Similarly

$$\lim_{x\to5^+}f(x) = \lim_{x\to5^+}\frac{(3x+1)(x-5)}{(2x+1)(x-5)} = \frac{16}{11}$$

It turns out it would have been sufficient to consider the full limit (and compute everything in one go) since it can be computed directly. With the information obtained from computing these limits we conclude that there is no vertical asymptote at $x = 5$. In this situation we actually have a hole.

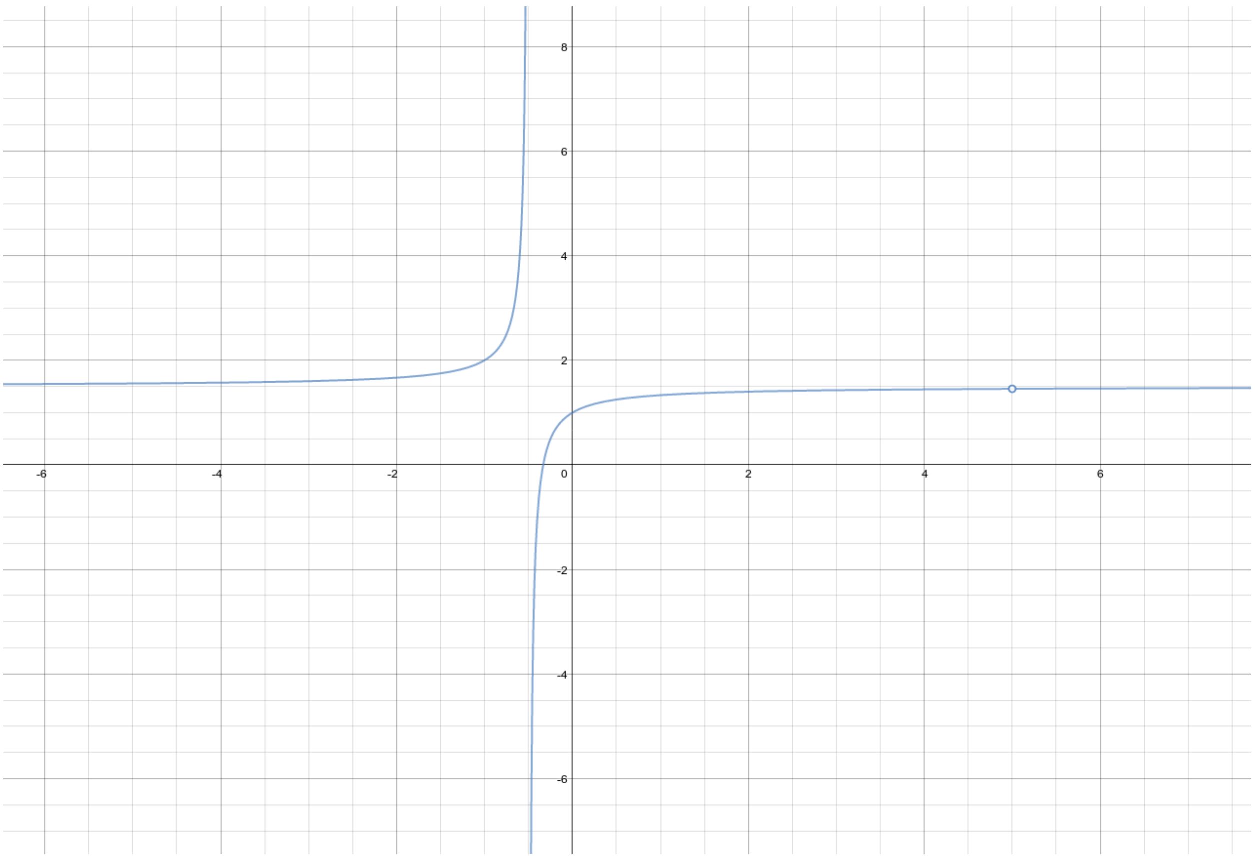
Let's now test our other candidate for vertical asymptote. Consider

$$\begin{aligned}\lim_{x\to-\frac{1}{2}^-}\frac{3x^2-14x-5}{2x^2-9x-5} &= \lim_{x\to-\frac{1}{2}^-}\frac{(3x+1)(x-5)}{(2x+1)(x-5)}\\ &= \lim_{x\to-\frac{1}{2}^-}\frac{3x+1}{2x+1}\\ &= \infty.\end{aligned}$$

We arrive at ∞ since for $x < -1/2$ the top and bottom are both negative. Similarly we compute the other one sided limit

$$\lim_{x \to -\frac{1}{2}^+} f(x) = \lim_{x \to -\frac{1}{2}^+} \frac{3x+1}{2x+1}$$
$$= -\infty.$$

We achieve $-\infty$ since for $x > -1/2$ the top is negative and the bottom is positive. Since at least one of the one sided limits comes out to $\pm\infty$ we therefore have a vertical asymptote with equation $x = -1/2$. By the way, here is a graph of the function $f(x)$. Observe the asymptotes and hole.



5. Consider the function

$$g(x) = \frac{\cos(3x)}{x}.$$

(a) Explain what happens to the numerator as x approaches ∞.

(b) Explain what happens to the denominator as x approaches ∞.

(c) Using your answers from (a) and (b) explain what happens to the values of $g(x)$ as $x \to \infty$. In this way you can suggest a value for

$$\lim_{x \to \infty} \frac{\cos(3x)}{x}.$$

(d) **Bonus:** How many times does $g(x)$ cross its horizontal asymptote? Explain how you know.

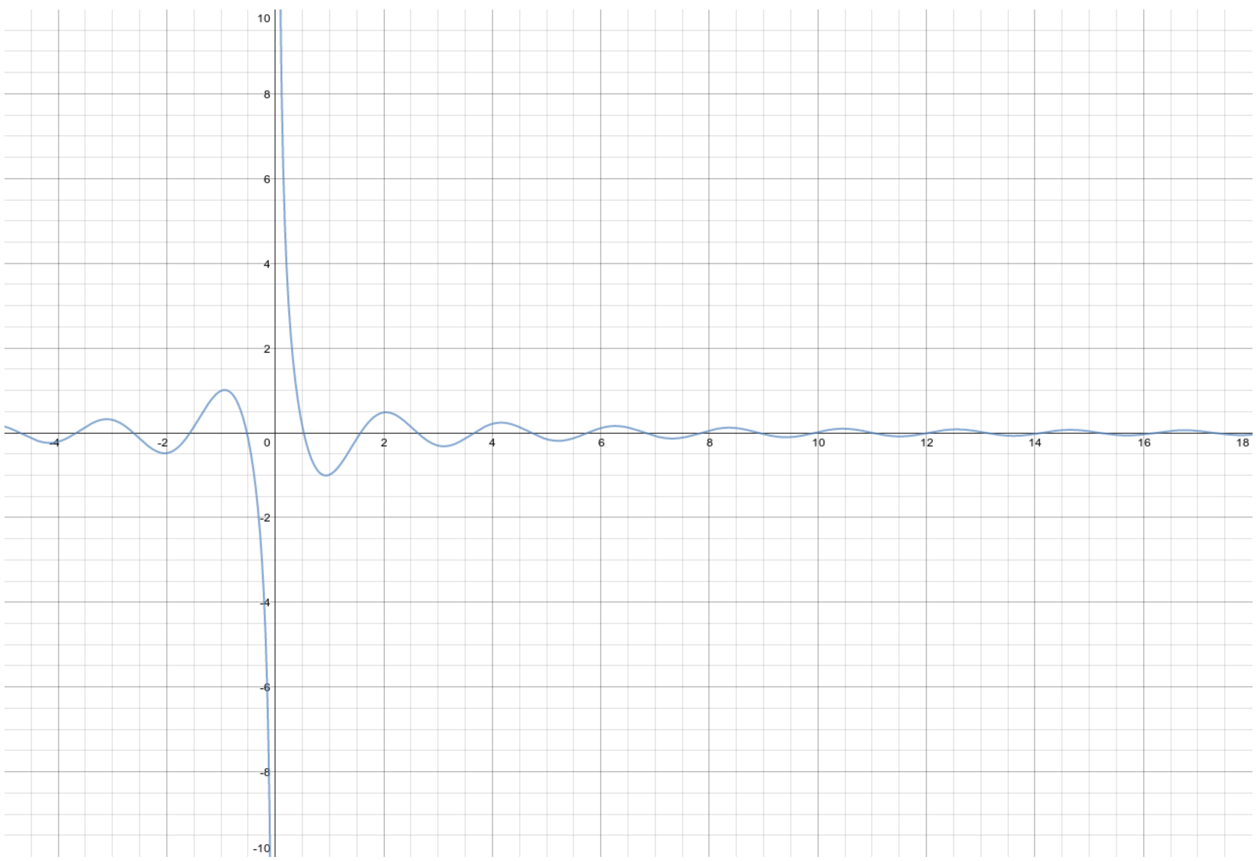
Solution: (a) As x becomes large the values of $\cos(3x)$ oscillate between 1 and -1. Note that the limit $\lim_{x\to\infty} \cos(3x)$ does not exist since there is no one value that the function approaches.
(b) As x becomes large the denominator (which is also x) becomes large. In this way the denominator approaches infinity. We could also say that as x becomes large $1/x$ becomes small. That is $\lim_{x\to\infty} \frac{1}{x} = 0$.
(c) Now we think about the whole function $g(x)$. For large values of x the numerator will be a number between -1 and 1. Dividing that number by the large x will yield a small number. Even though the values in the numerator are oscillating (and so the function is sometimes increasing) if we take x large enough we can make the values of $g(x)$ as small as we like. In this way we get that

$$\lim_{x\to\infty} \frac{\cos(3x)}{x} = 0$$

and so $g(x)$ has a horizontal asymptote at with equation $y = 0$.
(d) The function $g(x)$ will cross its horizontal asymptote $y = 0$ infinitely many times. We know this because the numerator $\cos(3x)$ will hit zero infinitely many times and so $g(x)$ has infinitely many zeros. We can find these values where $g(x) = 0$ by solving $\cos(3x) = 0$. This yields

$$x = \frac{\pi}{6} + n\frac{\pi}{3}$$

where n is an integer.
Observe (next page) the graph of function $g(x)$ and note the horizontal asymptote. Notice also that successive peaks take smaller and smaller values. In this way we can convince ourselves that $\lim_{x\to\infty} g(x) = 0$.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