Math 190 Homework 5: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:

1. Consider the function $f(x) = \dfrac{1}{x}$.
   (a) Find the slope of the secant line passing through the points $(2, 1/2)$ and $(1.5, 1/1.5)$ (feel free to use a calculator).
   (b) Find the slope of the secant line passing through the points $(2, 1/2)$ and $(1.9, 1/1.9)$.
   (c) Find the slope of the secant line passing through the points $(2, 1/2)$ and $(1.99, 1/1.99)$.
   (d) What do you think the slope of the tangent line at $(2, 1/2)$ is?

   Solution: In (a)-(c) we find the slope of the secant line using the formula

   $$m_{\text{sec}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

   (a)

   $$m_{\text{sec}} = \frac{1/2 - 1/1.5}{2 - 1.5} = -\frac{1}{3}$$

   (b)

   $$m_{\text{sec}} = \frac{1/2 - 1/1.9}{2 - 1.9} \approx -0.2631$$

   (c)

   $$m_{\text{sec}} = \frac{1/2 - 1/1.99}{2 - 1.99} \approx -0.2512$$

   (d) So, based on the slopes of the above secant lines, it looks like the slope of the tangent line is about -0.25.

2. Using a limit find the slope of the tangent line to $f(x) = 1/x$ at $(2, 1/2)$. Was your prediction in Question 1 correct?

   Solution:
   First, let's find the derivative of $f(x)$ using the limit definition

   $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

Also possible is to replace x by 2 in the above and compute from there. Computing

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\frac{x-(x+h)}{x(x+h)} \\
&= \lim_{h\to 0} \frac{1}{h}\frac{-h}{x(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+0)} \\
&= -\frac{1}{x^2}.
\end{aligned}$$

And so we achieve

$$f'(2) = -\frac{1}{4}$$

as predicted in Question 1.

3. Using the limit definition of the derivative (and not any other method) find the derivative of the function

$$f(x) = \frac{x}{x-1}$$

and use it to compute the slope of the tangent line to $f(x)$ at $x = 4$.

Solution: Let us start by computing the derivative using the limit definition. We find a common denominator, expand and simplify:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x+h}{x+h-1} - \frac{x}{x-1}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+h)(x-1) - x(x+h-1)}{(x-1)(x+h-1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x^2 - x + xh - h - x^2 - xh + x}{(x-1)(x+h-1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{(x-1)(x+h-1)}\right) \\
&= \lim_{h\to 0} \left(\frac{-1}{(x-1)(x+h-1)}\right) \\
&= \frac{-1}{(x-1)(x+0-1)} \\
&= \frac{-1}{(x-1)^2}.
\end{aligned}$$

Therefore, the slope of the tangent line to $f(x)$ at the point $x = 4$ is

$$f'(4) = -\frac{1}{(4-1)^2} = -\frac{1}{9}.$$