Math 190 Homework 6: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:

1. For the following problems find the derivative of the given function.

   (a) $2x^3 + \dfrac{3}{\sqrt{x}} + x^e - 4e^x$

   (b) $e^x \cos x$

   (c) $\dfrac{\cos x}{\sin x}$

   Solution:

   (a) $\left(2x^3 + \dfrac{3}{\sqrt{x}} + x^e - 4e^x\right)' = 6x^2 - \dfrac{3}{2}x^{-3/2} + ex^{e-1} - 4e^x$

   (b) $(e^x \cos x)' = e^x \cos x - e^x \sin x$

   (c) $\left(\dfrac{\cos x}{\sin x}\right)' = \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \dfrac{-1}{\sin^2 x}$ (simplification is not necessary)

2. Find the equation of the tangent line to

$$h(x) = \frac{x \cos x}{x+1}$$

   at the point $x = \pi$.

   Solution:

   We first find the derivative of $h(x)$ using product and quotient rule. Observe

$$\begin{aligned} h'(x) &= \frac{(\cos x - x \sin x)(x+1) - x\cos x}{(x+1)^2} \\ &= \frac{x\cos x + \cos x - x^2 \sin x - \sin x - x \cos x}{(x+1)^2} \\ &= \frac{\cos x - x^2 \sin x - \sin x}{(x+1)^2}. \end{aligned}$$

   We now find $h'(\pi)$ which will give the slope of the desired tangent line

$$h'(\pi) = \frac{\cos \pi - \pi^2 \sin \pi - \sin \pi}{(\pi+1)^2} = -\frac{1}{(\pi+1)^2}.$$

   We also need the y-value $h(\pi)$:

$$h(\pi) = -\frac{\pi}{\pi+1}.$$

With this information we can find the equation of the tangent line in slope-point form

$$y + \frac{\pi}{\pi + 1} = -\frac{1}{(\pi + 1)^2}(x - \pi).$$

And so, we have achieved the desired line.

3. (a) Use product rule to find the derivative of $\sin^2 x$.
   (b) Find all x-values where

$$f(x) = \sin^2 x - \cos x$$

   has horizontal tangent lines.

Solution: (a) Let us write our function as a product

$$\sin^2 x = \sin x \sin x.$$

We now apply product rule to see

$$\frac{d}{dx}(\sin x \sin x) = \cos x \sin x + \sin x \cos x = 2 \sin x \cos x.$$

(b) We seek horizontal tangent lines. Note that the slope of a horizontal tangent line is zero. Since the derivative gives the slope of the tangent line our strategy will be to find where all values where the derivative is zero. First we compute

$$f'(x) = 2 \sin x \cos x + \sin x$$

and then set $f'(x) = 0$. That is we wish to solve

$$2 \sin x \cos x + \sin x = 0.$$

With some factoring we can find the desired x values. That is

$$\sin x(2 \cos x + 1) = 0$$

and so we must solve

$$\sin x = 0$$

as well as

$$2 \cos x + 1 = 0$$

or rather

$$\cos x = -\frac{1}{2}.$$

For $x \in [0, 2\pi)$ the equation $\sin x = 0$ implies $x = 0, \pi$. On the whole real line we have

$$x = n\pi$$

where n is an integer. Now again for $x \in [0, 2\pi)$ the equation $\cos x = -1/2$ implies that $x = 2\pi/3$ and $x = 4\pi/3$. Now since the cosine function repeats every two π on the whole line we have the following solutions (adding as well our $\sin x = 0$ solutions)

$$\begin{aligned} x &= \frac{2\pi}{3} + n2\pi \\ x &= \frac{4\pi}{3} + n2\pi \\ x &= n\pi \end{aligned}$$

where n is an integer. Thus, we have established all x values where $h(x)$ has a horizontal tangent line.

4. (a) Recall the derivative of $e^x \cos x$ from Question 1(b). Using your answer, find the derivative of

$$e^x \cos x \sin x$$

using product rule once.

(b) In the same way you solved Question 4(a) use the product rule twice to prove in the following *triple product rule*:

$$\frac{d}{dx}(fgh) = \frac{df}{dx}gh + f\frac{dg}{dx}h + fg\frac{dh}{dx}.$$

Solution: (a) Let us use our solution from Question 1(b) together with one product rule. Behold

$$\begin{aligned} \frac{d}{dx}(e^x \cos x \sin x) &= \frac{d}{dx}(e^x \cos x) \sin x + e^x \cos x \cos x \\ &= (e^x \cos x - e^x \sin x) \sin x + e^x \cos^2 x \\ &= e^x \sin x \cos x - e^x \sin^2 x + e^x \cos^2 x. \end{aligned}$$

(b) We will prove the above triple product rule. Consider $g(x)h(x)$ as one function and apply product rule

$$\frac{d}{dx}\big(f(gh)\big) = \frac{df}{dx}(gh) + f\frac{d}{dx}(gh).$$

We now apply product rule a second time to see

$$\begin{aligned} \frac{d}{dx}\big(f(gh)\big) &= \frac{df}{dx}(gh) + f\frac{d}{dx}(gh) \\ &= \frac{df}{dx}gh + f\left(\frac{dg}{dx}h + g\frac{dh}{dx}\right) \\ &= \frac{df}{dx}gh + f\frac{dg}{dx}h + fg\frac{dh}{dx}. \end{aligned}$$

Therefore, we have achieved the desired triple product rule and so the proof is complete.

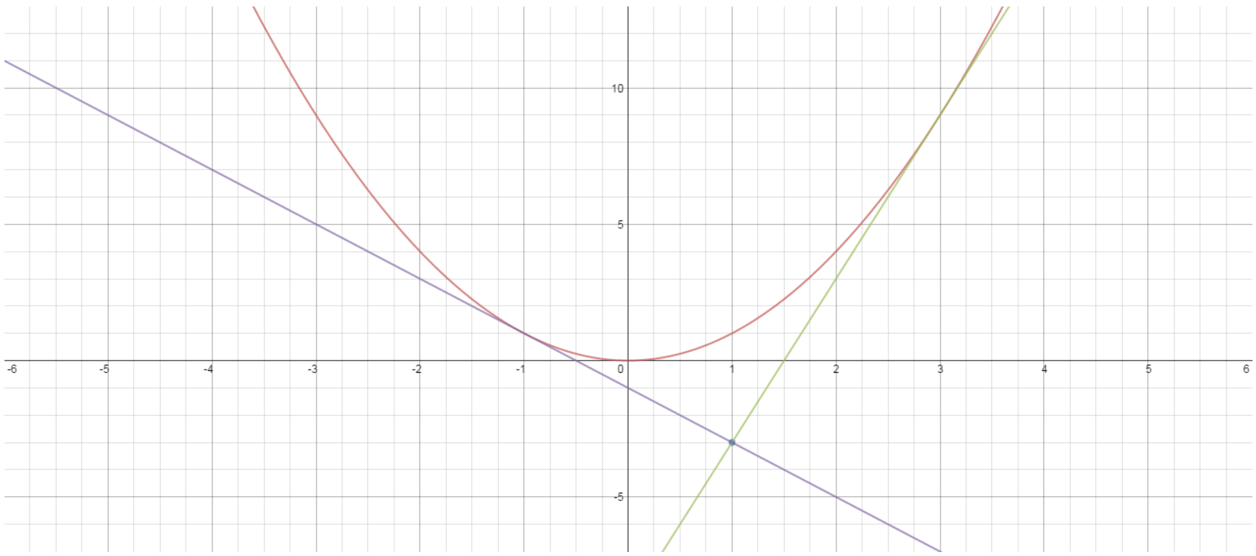
5. In this problem you will find all tangent lines of $f(x) = x^2$ which pass through the point $(1, -3)$. Consider following these steps:

   Step 1: Draw the graph of x^2 and label the point $(1, -3)$. Try to draw a line tangent to x^2 that passes through $(1, -3)$. Call this point p.

   Step 2: Find the equation of the tangent line to x^2 at point p.

   Step 3: Solve for the value(s) of p that will ensure your tangent line passes through $(1, -3)$.

   Solution: First we draw a careful picture.



Let us call the x values where our lines touch the parabola p. Let us find the equation of the tangent line at the point $x = p$. The slope of the tangent line is given by the derivative, which we can find

$$f'(x) = 2x$$

and so $f'(p) = 2p$. We also know that our tangent line passes through the point $(1, -3)$ so we can write the equation of our line in slope-point form

$$y + 3 = 2p(x - 1).$$

We also know that the point (p, p^2) must lay on the line and so we can substitute these values for (x, y). This gives

$$p^2 + 3 = 2p(p - 1)$$

which is a quadratic equation for p. We now solve this equation by factoring

$$\begin{aligned} 2p^2 - 2p - p^2 - 3 &= 0 \\ p^2 - 2p - 3 &= 0 \\ (p - 3)(p + 1) &= 0. \end{aligned}$$

Hence, we find the two desired values for p. That is $p = 3$ and $p = -1$. As we can see from our careful graph these values were expected (at the very least we should expect one positive and one negative value). Therefore, the equations of the two tangent lines are

$$\begin{aligned} y + 3 &= 6(x - 1) \\ y + 3 &= -2(x - 1). \end{aligned}$$

Which we could write instead as

$$\begin{aligned} y - 9 &= 6(x - 3) \\ y - 1 &= -2(x + 1). \end{aligned}$$

Alternatively, we could solve for p by noticing that the slope of the tangent lines is $m = 2p$ but also

$$m = \frac{p^2 - (-3)}{p - 1}$$

by using our rise/run. Setting these two m's equal to each other gives the same equation as we saw previously

$$\begin{aligned} 2p &= \frac{p^2 - (-3)}{p - 1} \\ 2p(p - 1) &= p^2 + 3. \end{aligned}$$