Math 190 Homework 8: Solutions

The assignment is due at the beginning of class on the due date. You are expected to provide full solutions, which are laid out in a linear coherent manner. Your work must be your own and must be self-contained. Your assignment must be stapled with your name and student number at the top of the first page.

Questions:

1. A hollow tree in the shape of a cylinder with radius 40 cm is filling with water during a rain storm. You notice that the hight of the water filling the tree is increasing at a rate of 10 cm/min. At what rate is the volume of water contained in the tree increasing when the height of the water is 3 meters.

   Solution: Let r, h, and V be the radius, height and volume of the water in the tree. The rate we have is dh/dt. The rate we require is dV/dt. Note that the radius of the water is not changing as the tree fills up so we can treat r as a constant. The formula for volume of a cylinder is

   $$V = \pi r^2 h.$$

   Let us take the derivative of both sides with respect to time to get

   $$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}.$$

   Note that r is not changing and so can be treated as a constant. Alternatively we could have used product rule and the fact that $dr/dt = 0$ to achieve the same formula. We can now substitute our known values for dh/dt and r to arrive at the desired rate.

   $$\frac{dV}{dt} = \pi (40 \text{ cm})^2 \cdot 10 \text{ cm/min} = \pi 40^2 \cdot 10 \text{ cm}^3/\text{min} \approx 50265 \text{ cm}^3/\text{min}.$$

   Note that this rate is the same no matter how high the water gets. In this way we did not need to use the value $h = 3$ m. We don't always use Chekhov's gun:

   `https://en.wikipedia.org/wiki/Chekhov%27s_gun`

2. A conical irrigation tank is being filled with water. The top of the tank has a radius of 2m and the tank is 6m high. If water is being pumped in at a rate of 4 m^3/s at what rate is the height of the water changing when the height is 3 m. Note that the volume of a cone is

   $$V = \frac{1}{3}\pi r^2 h$$

   where r is the radius and h is the height.

   Hint: This problem is more challenging because both r and h are functions of time. Consider using similar triangles to remove one of these variables.

   Solution: Consider the following picture which is meant to be a cross section of the cone. We
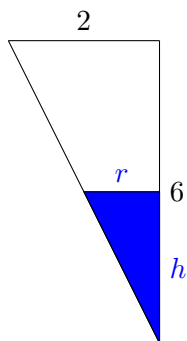
call the radius of the water r and the height of the water h; both of which are changing in time. The rate we have is dV/dt and the rate we require is dh/dt. The volume of a cone is

$$V = \frac{1}{3}\pi r^2 h.$$

Our equation has two variables. It will make this problem a lot easier to eliminate one of them. We will eliminate r in favour of h. Using similar triangle we achieve the following relationship between r and h

$$\frac{6}{2} = \frac{h}{r}$$

or in other words

$$r = \frac{h}{3}.$$

In this way our volume equation now reads

$$V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h$$

$$= \frac{1}{3^3}\pi h^3.$$

Now we take the derivative of both sides with respect to time and find

$$\frac{dV}{dt} = \frac{1}{3^3}\pi \cdot 3h^2 \frac{dh}{dt}$$

$$= \frac{1}{3^2}\pi h^2 \frac{dh}{dt}.$$

Rearranging gives the formula for the desired rate

$$\frac{dh}{dt} = \frac{3^2}{\pi h^2} \frac{dV}{dt}$$

after which we can substitute and find the value

$$\frac{dh}{dt} = \frac{3^2}{\pi(3 \text{ m})^2} \cdot 4 \text{ m}^3/\text{s}$$

$$= \frac{4}{\pi} \text{ m/s}.$$

3. Express the area of the half circle as an integral (we will compute this later in lecture).

   Solution: Consider the half circle centred at the origin with radius r. The equation of the circle is $x^2 + y^2 = r^2$. To write the circle as a function we solve for y:

   $$y = \sqrt{r^2 - x^2}.$$

   Note that we are taking the positive root since we want the top half of the circle (pictured). We
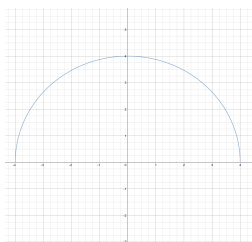
   seek the area under this curve from $x = -r$ to $x = r$. In light of this we can write the area of the half circle as the following integral

 $$\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx.$$

   We are not yet able to compute this integral but will be able to do so shortly. In this way we can find the equation for the area of the circle.

4. Consider the function

   $$f(x) = 2x + 1.$$

   (a) Find the area under the curve $f(x)$ on the interval $[0, 4]$ using the formula for area of a triangle and the formula for area of a rectangle.
   (b) Approximate this area using Riemann Sums. Use right endpoints and $n = 4$. Is your approximation an overestimate or underestimate? Explain why you expected this.

   Solution: (a) Let us get a picture of the required area So we can split this region into a rectangle
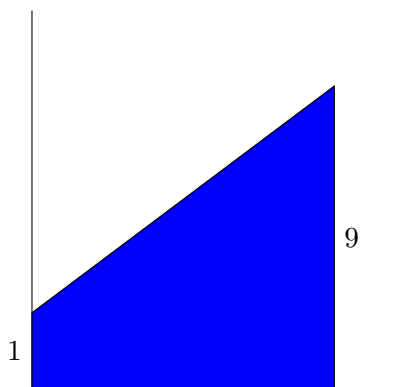

with length 4 and width 1 and a triangle with base 4 and height 8. Putting these two areas together gives the total area of the region

$$\int_0^4 (2x+1)dx = 1 \cdot 4 + \frac{1}{2} \cdot 4 \cdot 8 = 4 + 16 = 20.$$

Note that the area of a rectangle is length times width and the area of a triangle is half base times height.

(b) Now we will split the region into four sections and use rectangles (right endpoint) to approximate the area. Observe the following picture We seek the sum of the area of the four rectangles.
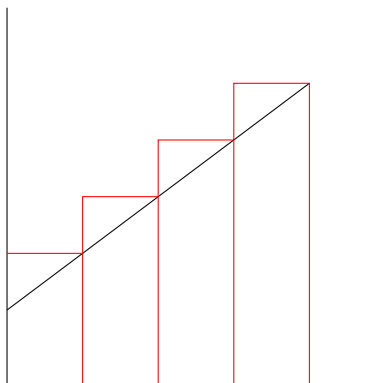
We can write this sum in the following way

$$\sum_{i=1}^{4} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$$

and

$$x_i = i.$$

In this way

$$\sum_{i=1}^{4} f(x_i)\Delta x = f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1$$

$$= 2(1) + 1 + 2(2) + 1 + 2(3) + 1 + 2(4) + 1$$

$$= 3 + 5 + 7 + 9 = 24.$$

This approximation is an overestimate. The true value is 20 (as calculated in part (a)). We expected this approximation to give us an overestimate from the start. Observing the graph we see that the area of the rectangles exceeds the desired area.

5. Consider again the function

$$f(x) = 2x + 1.$$

Compute the area under the curve (again on the interval $[0, 4]$) exactly by computing the following limit

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x.$$

You will need to use summation rules as well as the following summation formula

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Solution: First let us consider the sum for arbitrary n. We will only later take the limit as $n \to \infty$. First, since we have n bars the size of Δx is

$$\Delta x = \frac{b - a}{n} = \frac{4}{n}.$$

In light of this, the right endpoint for our first rectangle will be

$$x_1 = \frac{4}{n}$$

the second and third will be

$$x_2 = 2 \cdot \frac{4}{n}$$
$$x_3 = 3 \cdot \frac{4}{n}.$$

In general the right endpoint of the i'th rectangle will be

$$x_i = i \frac{4}{n}.$$

Therefore we can write our sum as

$$\sum_{i=1}^{n} f(x_i) \Delta x = \sum_{i=1}^{n} f\left(i \frac{4}{n}\right) \frac{4}{n} = \sum_{i=1}^{n} \left(2 \frac{4i}{n} + 1\right) \frac{4}{n}.$$

Using one of our summation rules, anything that doesn't depend on i can be factored out of the sum. So

$$\sum_{i=1}^{n} \left(2 \frac{4i}{n} + 1\right) \frac{4}{n} = \frac{4}{n} \sum_{i=1}^{n} \left(2 \frac{4i}{n} + 1\right).$$

Using another sum rule we can take the sum of each part separately

$$\frac{4}{n} \sum_{i=1}^{n} \left(2 \frac{4i}{n} + 1\right) = \frac{4}{n} \left(\sum_{i=1}^{n} 2 \frac{4i}{n} + \sum_{i=1}^{n} 1\right) = \frac{4}{n} \left(\frac{8}{n} \sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right)$$

and we have again factored some terms that are independent of i. Now we have a formula for the first sum. Namely

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

The second sum sould be equal to n. Observe

$$\sum_{i=1}^{n} = 1 + 1 + 1 + \ldots + 1 + 1 = n$$

since we have a total of n 1's. Putting these facts togeter gives

$$\sum_{i=1}^{n} f(x_i)\Delta x = \frac{4}{n}\left(\frac{8}{n}\frac{n(n+1)}{2} + n\right)$$

$$= \frac{16(n+1)}{n} + 4.$$

It remains to take the limit. We finally achieve

$$\int_{0}^{4} (2x+1)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \frac{16n+16}{n} + 4$$

$$= \lim_{n\to\infty} \frac{16 + 16/n}{1} + 4$$

$$= \frac{16}{1} + 4 = 20.$$

Observe that we got the same value in problem 4 (a). We can also verify this answer using the Fundamental Theorem of Calculus

$$\int_{0}^{4} (2x+1)dx = x^2 + x \Big|_{x=0}^{4} = 4^2 + 4 - 0^2 - 0 = 20.$$