Midterm Exam **Duration: 50 minutes**

*This test has **5 questions** on **6 pages**, for a total of 35 points.*

- Read all the questions carefully before starting to work.
- All questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ______________________ Last Name: ______________________

Student-No: ____

Signature: ____

Question:	1	2	3	4	5	Total
Points:	9	7	8	5	6	35
Score:						

Justify your answers and show all your work. Unless otherwise indicated simplification of answers is not required.

1. Compute the derivatives of the following functions

4 marks (a)

$$f(x) = \frac{4}{\sqrt{x}} + 2e^x + 7e^\pi + \frac{x^3x^2}{x^4}$$

Solution: First we write our function as the following

$$f(x) = 4x^{-1/2} + 2e^x + 7e^\pi + x$$

and then take the derivative using our power and exponential rules

$$f'(x) = -2x^{-3/2} + 2e^x + 0 + 1.$$

If you didn't notice the simplification of the final term (simplification is your friend *before* you take the derivative) you can use product and quotient rule as such

$$\begin{aligned}\left(\frac{x^3x^2}{x^4}\right)' &= \frac{(3x^2x^2 + 2xx^3)x^4 - 4x^3x^3x^2}{(x^4)^2} \\ &= \frac{5x^8 - 4x^8}{x^8} = 1\end{aligned}$$

5 marks (b)

$$g(x) = \sqrt{\ln(3x) + e^{\sin x}}$$

Solution: We use chain rule three times:

$$\begin{aligned}g'(x) &= \frac{1}{2}\left(\ln(3x) + e^{\sin x}\right)^{-1/2}\left(\ln(3x) + e^{\sin x}\right)' \\ &= \frac{1}{2}\left(\ln(3x) + e^{\sin x}\right)^{-1/2}\left(\frac{1}{3x}3 + e^{\sin x}\cos x\right) \\ &= \frac{1}{2}\left(\ln(3x) + e^{\sin x}\right)^{-1/2}\left(\frac{1}{x} + e^{\sin x}\cos x\right).\end{aligned}$$

2 marks

2. (a) Evaluate the following two quantities

$$\sin\left(\frac{2\pi}{3}\right) \text{ and } \cos\left(\frac{2\pi}{3}\right)$$

Solution: We compute using special triangles/unit circle/your preferred method

$$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

5 marks

(b) Find the equation of the tangent line to

$$f(x) = \frac{\cos x}{\sin x}$$

at the point $x = 2\pi/3$.

Solution: We start by finding the slope of the tangent line. For this we need the derivative. Apply now quotient rule to see

$$\begin{aligned} f'(x) &= \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} \\ &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x}. \end{aligned}$$

To find the slope of the tangent line at $x = 2\pi/3$ we compute

$$f'\left(\frac{2\pi}{3}\right) = -\frac{1}{\sin^2(2\pi/3)} = -\frac{4}{3}.$$

Now with the slope in hand our equation for the tangent line takes the form (using point-slope form)

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - y_1 &= -\frac{4}{3}(x - x_1). \end{aligned}$$

We have the point $x_1 = 2\pi/3$ and so we find

$$y_1 = f(2\pi/3) = \frac{\cos(2\pi/3)}{\sin(2\pi/3)} = -\frac{1}{\sqrt{3}}.$$

All together the equation of the desired tangent line is

$$y + \frac{1}{\sqrt{3}} = -\frac{4}{3}\left(x - \frac{2\pi}{3}\right).$$

3. Show all relevant limit computations.

3 marks (a) Find the equation(s) of all horizontal asymptotes of the following function

$$f(x) = \frac{5}{e^{-x}+7}.$$

Solution: To find the H.A. consider the following limits

$$\lim_{x\to\infty} \frac{5}{e^{-x}+7} = \frac{5}{0+7} = \frac{5}{7}$$

and

$$\lim_{x\to-\infty} \frac{5}{e^{-x}+7} = \frac{5}{\text{“}\infty\text{”}+7} = 0.$$

And so we have two H.A. with equations: $y = 5/7$ and $y = 0$.

5 marks (b) Find the equation(s) of all vertical asymptotes of the following function

$$g(x) = \frac{|x-2|}{x^2-2x-3}.$$

For each vertical asymptote determine whether each corresponding one sided limit equals $+\infty$ or $-\infty$.

Solution: We start by factoring the bottom to identify candidates for V.A.

$$x^2 - 2x - 3 = (x-3)(x+1)$$

and so our candidates are $x = 3$ and $x = -1$. We now compute all four one sided limits. In this way we establish if they are $+\infty$ or $-\infty$ or neither.

$$\lim_{x\to3^+} \frac{|x-2|}{(x-3)(x+1)} = +\infty \qquad \left(\frac{+}{(+)(+)}\right)$$

$$\lim_{x\to3^-} \frac{|x-2|}{(x-3)(x+1)} = -\infty \qquad \left(\frac{+}{(-)(+)}\right)$$

$$\lim_{x\to-1^+} \frac{|x-2|}{(x-3)(x+1)} = -\infty \qquad \left(\frac{+}{(-)(+)}\right)$$

$$\lim_{x\to-1^-} \frac{|x-2|}{(x-3)(x+1)} = +\infty \qquad \left(\frac{+}{(-)(-)}\right)$$

And so we have two vertical asymptotes with equations $x = 3$ and $x = -1$.

1 mark 4. (a) Given function $f(x)$ state the definition of its derivative.

Solution: The definition of the derivative is given by the following limit

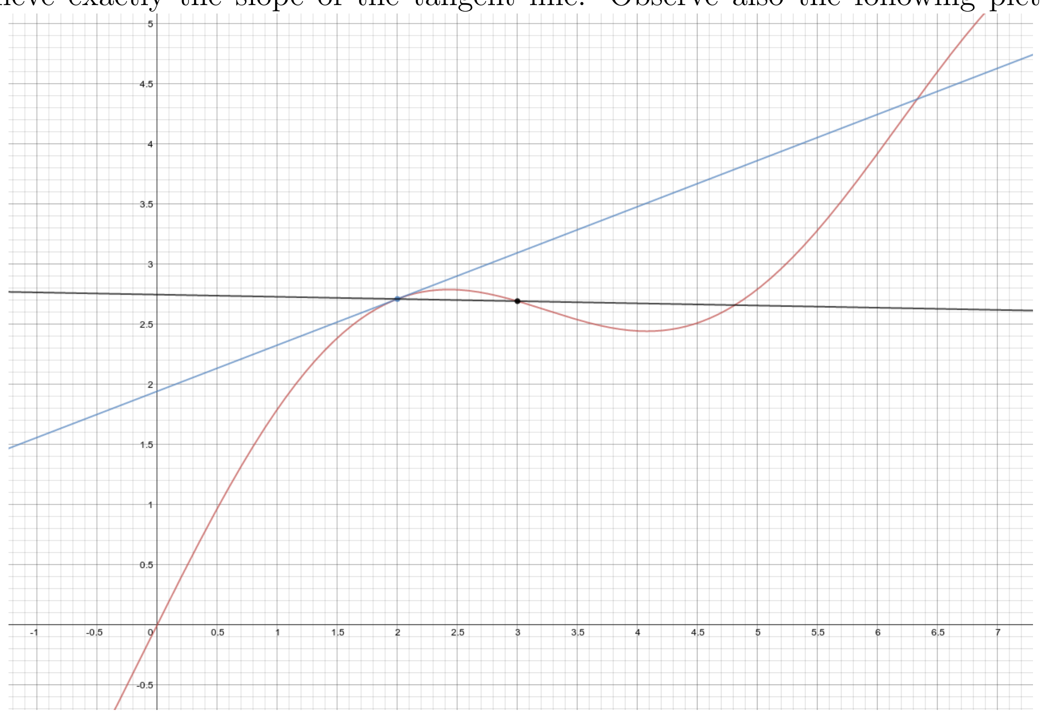
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

4 marks (b) Explain why your expression in part (a) should give you the slope of the tangent line to $f(x)$ at point x. Consider including a picture with your explanation.

Solution: The expression

$$\frac{f(x+h)-f(x)}{h}$$

(the so called difference quotient) is the slope of a secant line passing through points $(x, f(x))$ and $(x+h, f(x+h))$. For small values of h the slope of the secant line will be quite close to the slope of the tangent line. In the limit as $h \to 0$ we achieve exactly the slope of the tangent line. Observe also the following picture



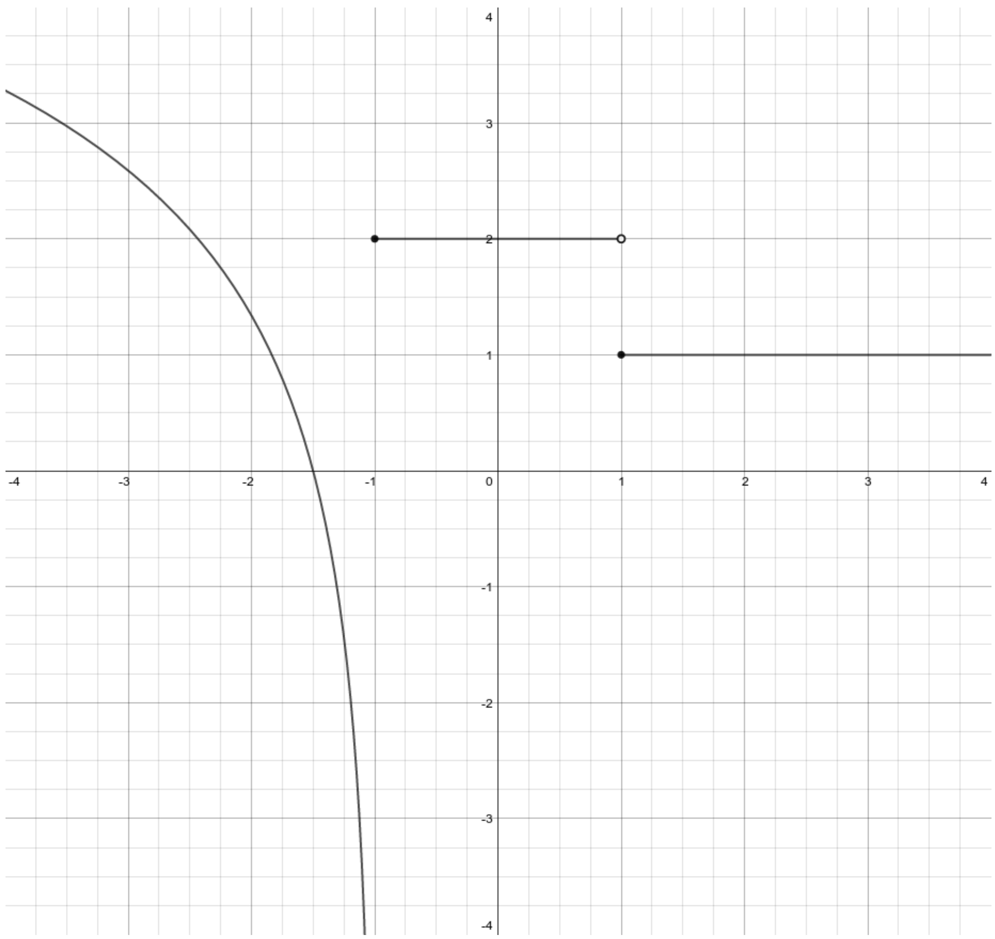
The slope of the secant line (black) is currently quite far from the slope of the tangent line (blue) but as the point $(x+h, f(x+h))$ (black) moves along the curve (red) the slope of the secant line will approach the slope of the tangent line.

6 marks 5. Sketch the graph of a function satisfying the following properties:

- The domain of $f(x)$ is $\{x \in \mathbb{R} : -4 \leq x \leq 4\}$
- $f(x)$ has a vertical asymptote at $x = -1$
- $\lim_{x \to -1^+} f(x) = 2$
- $\lim_{x \to 1} f(x)$ does not exist
- $f'(3) = 0$
- $f'(-3) = -1$

You do not need to find an equation for your function. Use the axes below.

Solution: There are many graphs that will satisfy the above six conditions. Here is an example of one such graph.



Observe that the slope at $x = 3$ is -1 and the slope at 3 is 0.