Science One Math

April 1, 2019

What we learned about power series so far…

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

- **Convergence** occurs on a symmetric interval centered at $x = a$,

(interval of convergence)

- The **sum** of a convergent power series $\sum c_n(x-a)^n$ is a function of x.

- We can **manipulate** power series to build functions.

Power series

Which of these are power series?

i) $\sum_{n=1}^{\infty} \ln(n)\,(2x+5)^n$

ii) $\sum_{n=0}^{\infty} n!\ (\pi - x)^n$

iii) $\sum_{n=1}^{\infty} \frac{(2n-1)!}{(2n)!}\ (x)^{2n+1}$

iv) $\sum_{n=1}^{\infty} \frac{(-2)^n \sin(n!x)}{n^2}$

A) *i* and *ii*

B) *ii* and *iii*

C) *i* and *iii*

D) *i, ii,* and *iii*

E) all of them

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E) all of them

Manipulating power series

Careful–some manipulations may change the interval of convergence.

- Making a change of variable → shifts the centre of the series
- Multiplying by a factor → changes interval of convergence

- Adding and subtracting
- Multiplying and dividing

} may change interval of convergence

- Differentiate term by term
- Integrate term by term

} does not change interval of convergence (may change convergence at endpoints)

Manipulating power series

- Making a change of variable
- Multiplying by a factor

e.g. $\frac{4x^{12}}{1+3x} = 4x^{12} \cdot \frac{1}{1-(-3x)} = 4x^{12}(1 + (-3x) + (-3x)^2 + (-3x)^3 + \cdots)$

$$\frac{4x^{12}}{1+3x} = \sum_{n=0}^{\infty} 4x^{12}(-3x)^n = \sum_{n=0}^{\infty} (-1)^n 4 \cdot 3^n x^{12+n}$$

Gallery of functions we expressed as power series using some manipulations of $\sum_{n=0}^{\infty} x^n$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$

$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$ for $-1 < x < 1$

rational, logarithmic, inverse trig. funct.

$\frac{3x}{2-x} = \sum_{n=0}^{\infty} \frac{3}{2^{n+1}} x^{n+1}$ for $|x| < 2$

...what about e^x, $\sin(x)$, $\cos(x)$?

$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$ for $|x| < 1$

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ for $-1 < x \leq 1$

$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $-1 \leq x \leq 1$

Most functions can be produced by manipulating $\sum x^n$ ...except...

... except important functions like e^x, $\sin(x)$, $\cos(x)$.

$\Rightarrow$ need a different strategy to build appropriate coefficients c_n.

More generally, we are interested in the following questions:

- what is the power series representation of a function?
- which functions have power series representations?

Observation: the coefficients of a power series follow a pattern!

$f(x) = \frac{1}{1-x}$

$f'(x) = (1-x)^{-2}$ $\quad f''(x) = 2(1-x)^{-3}$ $\quad f'''(x) = 6(1-x)^{-4}$

$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 2, \quad f'''(0) = 6$

$$1 + x + x^2 + x^3 + \cdots = f(0) + f'(0) \cdot x + \frac{f''(0)}{2} x^2 + \frac{f'''(0)}{6} x^3 + \cdots$$

same coefficients as in the Taylor polynomials!

Taylor Polynomials

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2} + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Note $T_n(x)$ is the *partial sum* of the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$.

Recall, a series converges to S if $\lim_{n\to\infty} S_n = S$, where S_n are the partial sums.

We want $\lim_{n\to\infty} T_n(x) = f(x)$.

Recall, the error in approximating $f(x)$ with $T_n(x)$ is

$$f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$ for some c between a and x.

If we let the degree n to go to infinity, does the error go to zero? *Yes, depends on f and x*

Example: Let $f(x) = e^x$. Does $R_n(x)$ approach zero for $n \to \infty$?

Where $R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

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Answer:

- Build the Taylor polynomial at $a = 0$, $T_n(x) = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n$
- Then $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + R_n(x)$.
- Now fix x, then $|R_n(x)| = \frac{e^c}{(n+1)!}|x|^{n+1} \leq M\frac{|x|^{n+1}}{(n+1)!}$ for some constant M>0
- observe $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = \lim_{n\to\infty} \frac{|x|}{(n+1)} \frac{|x|}{n} \ldots \frac{|x|}{2} \frac{|x|}{1} = 0$ for any x
- Then by squeeze theorem $\lim_{n\to\infty} |R_n(x)| = 0$

$\Longrightarrow \lim_{n\to\infty} T_n(x) = e^x$, that is $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ converge to e^x for any x

Taylor series

a power series representation of a function

Theorem: If f has a power series representation at *a*, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

then the sequence generating the coefficients of the series is

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

The series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the **Taylor series** of f at *a.*

f is called *analytic* on the convergence interval of its Taylor series.

Common Maclaurin series (Taylor series centred at 0)

- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$
- $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ for all x
- $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}x^{2n+1}$ for all x
- $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}x^{2n}$ for all x
- $\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1}x^{n+1}$ for $-1 < x \leq 1$
- $\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)}x^{2n+1}$ $-1 \leq x \leq 1$

Recall: operations on power series (provided it converges)

- Changing variable
- Multiplying by a factor
- Differentiating term by term
- Integrating term by term

Problem: Find the first few terms of the Taylor series centred at $x = 0$ of $e^{\sin x}$.

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- Changing variable
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Problem: Find the first few terms of the Taylor series centred at $x = 0$ of $e^{\sin x}$.

Recall $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$ and $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$

By substitution,

$$e^{\sin x} = 1 + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right) + \frac{1}{2}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right)^2 + \cdots =$$

$$[combine\ like\ terms] \qquad = 1 + x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 - \frac{1}{3!}x^4 + \frac{1}{5!}x^5 \ldots$$