Science One Math

April 3, 2019

Taylor series

If f has a power series representation $\sum_{n=0}^{\infty} c_n (x-a)^n$ then $\boxed{c_n = \frac{f^{(n)}(a)}{n!}.}$

We want the series to converge to $f(x)$.

Taylor series

a power series representation of a function

Theorem: If f has a power series representation at *a*, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

then the sequence generating the coefficients of the series is

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

The series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the **Taylor series** of f at *a.*

f is called *analytic* on the convergence interval of its Taylor series.

Some common Maclaurin series (Taylor series centred at 0)

- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ $\quad -1 < x < 1$
- $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ $\quad$ all x
- $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}x^{2n+1}$ all x
- $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}x^{2n}$ $\quad$ all x
- $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1}x^{n+1}$ $\quad -1 < x \leq 1$
- $\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)}x^{2n+1}$ $-1 \leq x \leq 1$

Recall: operations on power series (provided it converges)

- Changing variable
- Multiplying and adding series
- Differentiating term by term
- Integrating term by term

E.g. Find the MacLaurin series for $f(x) = x^3 e^{-3x^2}$.

$$x^3 e^{-3x^2} = x^3 \sum_{n=0}^{\infty} \frac{(-3x^2)^n}{n!} = \sum_{n=0}^{\infty} x^3 \frac{(-3)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^{2n+3}$$

Stuff you can do with Taylor Series: compute integrals

(April 2018) Suppose the distance $R \geq 0$ of a quantum particle from a certain point is a random variable described by the probability density function $f(r) = \frac{2}{\sqrt{\pi}} e^{-r^2}$

Write an integral giving the probability that the particle is a distance no more than 1 from the point, and express it as an infinite series expression.

Stuff you can do with Taylor Series: compute integrals

(April 2018) Suppose the distance $R \geq 0$ of a quantum particle from a certain point is a random variable described by the probability density function

$$f(r) = \frac{2}{\sqrt{\pi}} e^{-r^2}$$

Write an integral giving the probability that the particle is a distance no more than 1 from the point, and express it as an infinite series expression.

$$P(0 < R < 1) = \int_0^1 \frac{2}{\sqrt{\pi}} e^{-r^2} dr = \int_0^1 \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n (r)^{2n} dr =$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 r^{2n} dr = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{r^{2n+1}}{2n+1} \Bigg|_{r=0}^{r=1} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2n+1}$$

Stuff you can do with Taylor Series: compute integrals

Problem: (Final 2016) Find the Taylor series centred at $x = 0$ of

$$J(x) = \int_0^x \frac{e^t - 1}{t} dt.$$

$$\lim_{z\to 0^+} \int_z^x \frac{e^t - 1}{t} dt = \lim_{z\to 0^+} \int_z^x \frac{\sum_{n=0}^{\infty} \frac{1}{n!} t^n - 1}{t} dt = \lim_{z\to 0^+} \sum_{n=1}^{\infty} \int_z^x \frac{1}{n!} t^{n-1} dt =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!n} x^n - \lim_{z\to 0^+} \sum_{n=1}^{\infty} \frac{1}{n!n} z^n = \sum_{n=1}^{\infty} \frac{1}{n!n} x^n$$

Stuff you can do with Taylor Series: compute integrals

The probability density function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ describes the **normal distribution (with mean 0 and standard deviation 1)**.

Problem: Find the probability that a normally distributed random variable lies *within* one standard deviation of the mean.

Stuff you can do with Taylor Series: compute integrals

The probability density function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ describes the **normal distribution (with mean 0 and standard deviation 1)**.

Problem: Find the probability that a normally distributed random variable lies within one standard deviation of the mean.

$$P(-1 < X < 1) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\frac{x^2}{2}} dx = \sqrt{\frac{2}{\pi}} \int_0^1 \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{x^2}{2}\right)^n dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n} (x)^{2n}\, dx = \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n} \int_0^1 x^{2n} dx$$

$$= \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n} \left.\frac{x^{2n+1}}{2n+1}\right|_0^1 = \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n\,(2n+1)} = \sqrt{\frac{2}{\pi}} \left(1 - \frac{1}{6} + \frac{1}{40} - \cdots\right)$$

$\sqrt{\frac{2}{\pi}} \approx 0.80,$ $\sqrt{\frac{2}{\pi}}\left(1 - \frac{1}{6}\right) \approx 0.66,$ $\sqrt{\frac{2}{\pi}}\left(1 - \frac{1}{6} + \frac{1}{40}\right) \approx \mathbf{0.68}$

Stuff you can do with Taylor Series: compute sums of series of numbers

Problem: (Final 2017) Find the sum of the series

$$1+\frac{1}{2}+\left(\frac{1}{2}\right)^2\frac{1}{2!}+\left(\frac{1}{2}\right)^3\frac{1}{3!}+\left(\frac{1}{2}\right)^4\frac{1}{4!}+\cdots+\left(\frac{1}{2}\right)^n\frac{1}{n!}\ldots$$

(part b) Find the sum of $1+\frac{1}{2}\cdot 2+\frac{1}{4}\cdot 3+\cdots+\frac{n}{2^{n-1}}+\cdots$

Stuff you can do with Taylor Series: compute sums of series of numbers

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Recall $\quad e^x=1+x+\frac{1}{2}x^2+\frac{1}{3!}x^3+\cdots=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$

Then $e^{1/2}=1+\frac{1}{2}+\left(\frac{1}{2}\right)^2\frac{1}{2!}+\left(\frac{1}{2}\right)^3\frac{1}{3!}+\left(\frac{1}{2}\right)^4\frac{1}{4!}+\cdots$

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Stuff you can do with Taylor Series: compute sums of series of numbers

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$$1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 \frac{1}{2!} + \left(\tfrac{1}{2}\right)^3 \frac{1}{3!} + \left(\tfrac{1}{2}\right)^4 \frac{1}{4!} + \cdots + \left(\tfrac{1}{2}\right)^n \frac{1}{n!} \ldots$$

Recall $\quad e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

Then $\quad e^{1/2} = 1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 \frac{1}{2!} + \left(\tfrac{1}{2}\right)^3 \frac{1}{3!} + \left(\tfrac{1}{2}\right)^4 \frac{1}{4!} + \cdots$

(part b) Find the sum of $1 + \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 3 + \cdots + \frac{n}{2^{n-1}} + \cdots$

$\sum_{n=1}^{\infty} n(x)^{n-1} = \frac{d}{dx} \sum_{n=0}^{\infty} (x)^n = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$ eval. at $x = \frac{1}{2}$

Stuff you can do with Taylor Series: compute π

Compute π using the sum of $\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)} x^{2n+1}$ for $x = 1$.

We know $\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)} x^{2n+1}$ converges for $-1 \leq x \leq 1$.

Then, for $x = 1$

$$\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)} = \arctan(1) = \frac{\pi}{4}$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

Slow convergence (200 terms to get value correct to 2 decimal digits!).

Faster to compute $\pi = 6\arctan\left(\frac{1}{\sqrt{3}}\right) = 6\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)} \frac{1}{\left(\sqrt{3}\right)^{2n+1}}$

(can get 6 decimal places with ten terms!)

The algebra of convergent series

Can a convergent series be manipulated as a finite sum?

Yes, if it converges absolutely, otherwise no!

The delicacy of conditionally convergent series

If a series **converges only conditionally**, the order of the terms is important.

$$\log(1+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} \cdots = \ln 2$$

Rearrange

$$\left(1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} \cdots\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{12} \cdots\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{20} \cdots\right)$$

$$\underbrace{\left(1 - \sum \left(\tfrac{1}{2}\right)^n\right)}_{\to 0} + \frac{1}{3}\underbrace{\left(1 - \sum \left(\tfrac{1}{2}\right)^n\right)}_{\to 0} + \frac{1}{5}\underbrace{\left(1 - \sum \left(\tfrac{1}{2}\right)^n\right)}_{\to 0} + \cdots \quad \text{we get} \quad 0 \neq \ln 2$$

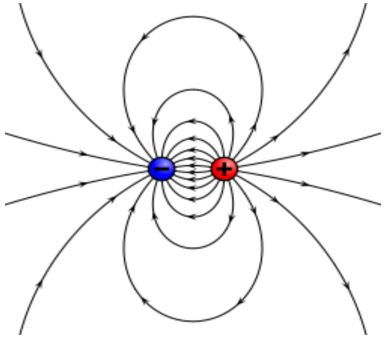
cannot rearrange the order of terms

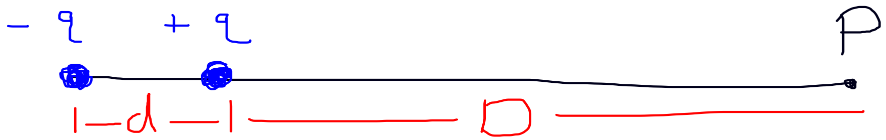
Stuff you can do with Taylor Series: make approximations

Approximate the E-field at distance $D \gg d$ from a dipole

- $E = \frac{kq}{D^2} - \frac{kq}{(D+d)^2} = \frac{kq}{D^2}\left(1 - \frac{1}{\left(1+\frac{d}{D}\right)^2}\right)$
- $\frac{1}{(1+x)^2} = -\frac{d}{dx}\frac{1}{1+x} = -\frac{d}{dx}(1 - x + x^2 - x^3 + \cdots) = 1 - 2x + 3x^2 - \cdots$
- $E = \frac{kq}{D^2}\left(1 - \left[1 - 2\frac{d}{D} + 3\left(\frac{d}{D}\right)^2 - \cdots\right]\right) = \frac{2kqd}{D^2} - \frac{3kqd^2}{D^4} + \cdots$

Stuff you can do with Taylor series: make approximations





Approximate the E-field at distance $D \gg d$ from a dipole (figure):

- $E = \frac{kq}{D^2} - \frac{kq}{(D+d)^2} = \frac{kq}{D^2}\left(1 - \frac{1}{\left(1+\frac{d}{D}\right)^2}\right)$
- $\frac{1}{(1+x)^2} = -\frac{d}{dx}\frac{1}{1+x} = -\frac{d}{dx}\left(1 - x + x^2 - x^3 + \cdots\right)$
  $= 1 - 2x + 3x^2 - \cdots$
- $E = \frac{kq}{D^2}\left(1 - \left[1 - 2\frac{d}{D} + 3\left(\frac{d}{D}\right)^2 - \cdots\right]\right) = \boxed{\frac{2kqd}{D^3} - \frac{3kqd^2}{D^4} + \cdots}$

Yet more stuff you can do with Taylor series

(…this is not going to be on the exam...)

Stuff you can do with Taylor Series: solve ODEs

Solve the ODE initial value problem : $y' + y = x, \quad y(0) = 0.$

- Try power series $\quad y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$
- Then $\quad y' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$
- Solve $\quad x = y' + y = (c_1 + c_0) + (2c_2 + c_1)x + (3c_3 + c_2)x^2 + \cdots$
- So choose $\quad c_1 + c_0 = 0, \quad 2c_2 + c_1 = 1, \quad 3c_3 + c_2 = 0, \ldots$
- So $\quad c_1 = -c_0, \;\; c_2 = \frac{1}{2} - \frac{c_1}{2} = \frac{1}{2} + \frac{c_0}{2}, \quad c_3 = -\frac{c_2}{3} = -\frac{1}{6} - \frac{c_0}{6}$
- $y(0) = 0 \implies c_0 = 0$, so $c_1 = 0, c_2 = \frac{1}{2}, c_3 = -\frac{1}{6}, \ldots$

  $\implies y(x) = \frac{1}{2} x^2 - \frac{1}{6} x^3 + \ldots$

(actual solution $y(x) = x - 1 + e^{-x}$)

Stuff you can do with Taylor Series: e^{ix}

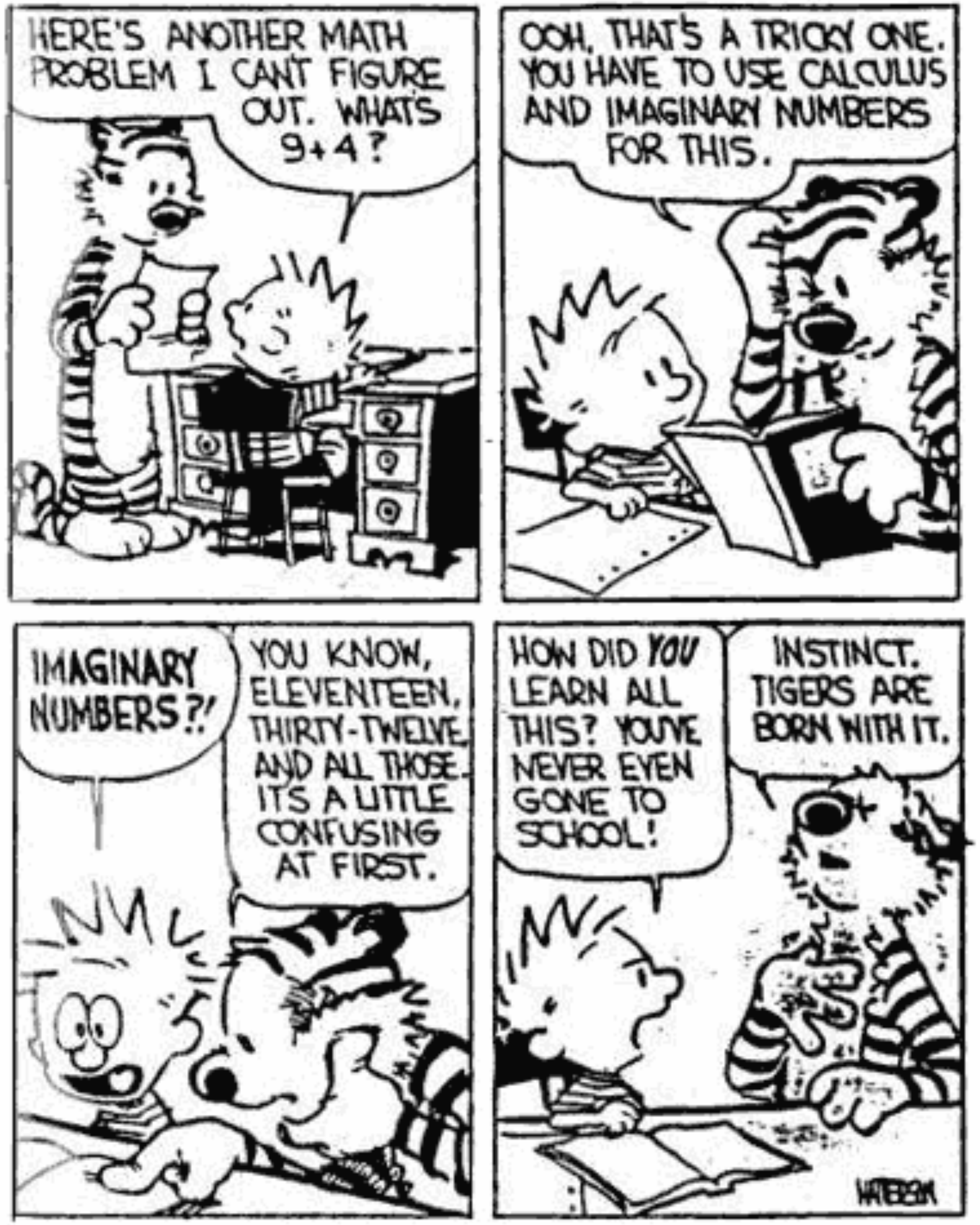
- Complex numbers: $z = a + ib$. How does i work?
- $\boxed{i^2 = -1}$, $i^3 = (-1)i = -i$, $i^4 = (-i)i = 1$, $i^5 = i$, ...
- $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$
- $e^{ix} = 1 + ix + \frac{1}{2}i^2x^2 + \frac{1}{3!}i^3x^3 + \frac{1}{4!}i^4x^4 + \frac{1}{5!}i^5x^5 + \cdots$

$$= 1 + ix - \frac{1}{2}x^2 - \frac{1}{3!}ix^3 + \frac{1}{4!}x^4 + \frac{1}{5!}ix^5 + \cdots$$

$$= \left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots\right) + i\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 -\right)$$

$$= (\quad \cos x \quad) + i(\quad \sin x \quad)$$

- Conclusion $\boxed{e^{ix} = \cos(x) + i\sin(x)}$ <u>*Euler's formula*</u>


HERE'S ANOTHER MATH PROBLEM I CAN'T FIGURE OUT. WHAT'S 9+4?
OOH, THAT'S A TRICKY ONE. YOU HAVE TO USE CALCULUS AND IMAGINARY NUMBERS FOR THIS.
IMAGINARY NUMBERS?!
YOU KNOW, ELEVENTEEN, THIRTY-TWELVE AND ALL THOSE. IT'S A LITTLE CONFUSING AT FIRST.
HOW DID YOU LEARN ALL THIS? YOU'VE NEVER EVEN GONE TO SCHOOL!
INSTINCT. TIGERS ARE BORN WITH IT.
WATTERSON

The end