More Improper Integrals

. . . last time . . .

Improper Integrals (Type I: infinite integration domain)

$$\int_a^\infty f(x)dx = \lim_{R\to\infty} \int_a^R f(x)dx \quad \text{if this limit exists}$$

$$\int_{-\infty}^b f(x)dx = \lim_{r\to-\infty} \int_r^b f(x)dx \quad \text{if this limit exists}$$

$$\int_{-\infty}^\infty f(x)dx = \lim_{r\to-\infty} \int_r^c f(x)dx + \lim_{R\to\infty} \int_c^R f(x)dx$$

if **both** these limits exist

Example: what is tricky about $\int_0^1 \ln(x)dx$?

The integrand $\ln(x)$ is continuous for $0 < x \leq 1$, but *not* at the left endpoint $x = 0$, so:

- we first compute, for any $0 < t < 1$,

$$\int_t^1 \ln(x)dx = x\ln(x)|_t^1 - \int_t^1 dx = -t\ln(t) - (1-t)$$

(using integration by parts!)

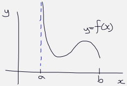
- then we send $t \to 0$ (from the right):

$$\int_0^1 \ln(x)dx = \lim_{t\to 0+} \int_t^1 \ln(x)dx = \lim_{t\to 0+} (-t\ln(t) - 1 + t) = \boxed{-1}$$

(since $\lim_{t\to 0+} t\ln(t) = 0$ — why?)

Improper Integrals (Type II: unbounded integrand)

- f continuous on $(a, b]$, but not at $x = a$:



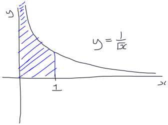
$$\int_a^b f(x) := \lim_{t \to a+} \int_t^b f(x)dx \quad \text{if this limit exists}$$

- f continuous on $[a, b)$, but not at $x = b$:

$$\int_a^b f(x) := \lim_{T \to b-} \int_a^T f(x)dx \quad \text{if this limit exists}$$

- if the limit exists, the improper integral **converges**. If not, it **diverges**.

Find the area of the region under $y = \frac{1}{\sqrt{x}}$, $0 < x \leq 1$:

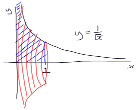


$$A = \int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{t \to 0+} \int_t^1 \frac{1}{\sqrt{x}} dx$$

$$= \lim_{t \to 0+} 2\sqrt{x}\Big|_t^1 = 2 \lim_{t \to 0+} (1 - \sqrt{t}) = \boxed{2}$$

(this improper integral *converges*).

Find the volume obtained by rotating this region about the x-axis:



$$V = \int_0^1 \pi \left(\frac{1}{\sqrt{x}} \right)^2 dx = \pi \int_0^1 \frac{1}{x} dx = \pi \lim_{t \to 0+} \int_t^1 \frac{1}{x} dx$$

$$= \pi \lim_{t \to 0+} \ln(x)\Big|_t^1 = \pi \lim_{t \to 0+} (0 - \ln(t)) = \boxed{\infty}$$

(this improper integral *diverges*).

Examples:

1. $\int_0^1 (1-x)^{-\frac{1}{3}} dx = \lim_{T\to 1-} \int_0^T (1-x)^{-\frac{1}{3}} dx$

$$= \lim_{T\to 1-} -\frac{3}{2}(1-x)^{\frac{2}{3}}\Big|_0^T = \frac{3}{2}\lim_{T\to 1-}\left[-(1-T)^{\frac{2}{3}}+1\right] = \frac{3}{2}$$

2. $\int_0^{\pi/2} \cot(y)dy = \lim_{t\to 0+} \int_t^{\frac{\pi}{2}} \frac{\cos(y)}{\sin(y)} dy = \lim_{t\to 0+} \ln(\sin(y))\Big|_t^{\frac{\pi}{2}}$

$$= \lim_{t\to 0+} -\ln(\sin(t)) = \infty \text{ (diverges)}$$

3. $\int_0^{\pi} \sec^2(t)dt = \lim_{T\to\frac{\pi}{2}-} \int_0^T \sec^2(t)dt + \lim_{t\to\frac{\pi}{2}+} \int_t^{\pi} \sec^2(t)dt$

and $\lim_{T\to\frac{\pi}{2}-} \int_0^T \sec^2(t)dt = \lim_{T\to\frac{\pi}{2}-} \tan(T) = \infty$ (so diverges)

4. $\int_0^{\infty} e^{-x^2} dx$ ... stay tuned ...

Type I: for $\int_a^\infty f(x)dx$ to converge, $f(x)$ must go to zero as $x \to \infty$ fast enough: for example,

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{R\to\infty} \left\{ \begin{array}{ll} \frac{x^{1-p}}{1-p}\Big|_1^R = \frac{R^{1-p}-1}{1-p} & p \neq 1 \\ \ln(x)\Big|_1^R = \ln(R) & p = 1 \end{array} \right\} \left\{ \begin{array}{ll} < \infty & p > 1 \\ \infty & p \leq 1 \end{array} \right.$$

Type II: if $f(x)$ is unbounded at $x = 0$ (say), for $\int_0^b f(x)dx$ to converge, $f(x)$ must not go to infinity too quickly as $x \to 0+$:

$$\int_0^1 \frac{1}{x^p}dx = \lim_{t\to 0+} \left\{ \begin{array}{ll} \frac{x^{1-p}}{1-p}\Big|_t^1 = \frac{1-t^{1-p}}{1-p} & p \neq 1 \\ \ln(x)\Big|_t^1 = -\ln(t) & p = 1 \end{array} \right\} \left\{ \begin{array}{ll} < \infty & p < 1 \\ \infty & p \geq 1 \end{array} \right.$$

Notice: in both cases, $p = 1$ (i.e. the function $1/x$) is the borderline between convergent and divergent behaviour.

We can use knowledge of simple improper integrals to determine if more complicated improper integrals converge or diverge:

Comparison Theorem: if f and g are continuous on (a, b) (possibly $a = -\infty$ and/or $b = \infty$) with $0 \le f(x) \le g(x)$:

- if $\int_a^b g(x)dx$ **converges**, then $\int_a^b f(x)dx$ also **converges**
- if $\int_a^b f(x)dx$ **diverges**, then $\int_a^b g(x)dx$ also **diverges**

Example: does $\int_0^\infty e^{-x^2} dx$ converge or diverge?

- $0 \le e^{-x^2} \le e^{-x}$ for $x \ge 1$
- split: $\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx$
- the first integral is convergent (it is not even improper!)
- the second one, we compare with

$$\int_1^\infty e^{-x} dx = \lim_{R\to\infty} \int_1^R e^{-x} dx = \lim_{R\to\infty} (-e^{-R} + e^{-1}) = e^{-1} < \infty$$

- conclusion: $\int_0^\infty e^{-x^2} dx$ converges (actual value: $\sqrt{\pi}/2$)

Examples: does the integral converge or diverge?

1. $\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sqrt{x}} dx$ since $0 \leq \frac{\cos(x)}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$ for $0 \leq x \leq \frac{\pi}{2}$, and $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{x}}$ converges, this integral converges.

2. $\int_0^1 \frac{e^x}{x} dx$ since $\frac{e^x}{x} \geq \frac{1}{x}$ for $0 \leq x \leq 1$, and $\int_0^1 \frac{dx}{x}$ diverges, this integral diverges.

3. $\int_0^\infty \frac{dx}{\sqrt{x+x^3}}$ $= \int_0^1 \frac{dx}{\sqrt{x+x^3}} + \int_1^\infty \frac{dx}{\sqrt{x+x^3}}$. For $0 \leq x \leq 1$, $0 \leq \frac{1}{\sqrt{x+x^3}} \leq \frac{1}{\sqrt{x}}$, and $\int_0^1 \frac{dx}{\sqrt{x}}$ converges. For $1 \leq x < \infty$, $0 \leq \frac{1}{\sqrt{x+x^3}} \leq x^{-\frac{3}{2}}$, and $\int_1^\infty x^{-\frac{3}{2}} dx$ converges. So the integral converges.

4. $\int_1^\infty \frac{\sin(x)}{x} dx$ Tricky! Integrate by parts: $\int_1^\infty \frac{\sin(x)}{x} dx = -\frac{\cos(x)}{x}\Big|_1^\infty - \int_1^\infty \frac{\cos(x)}{x^2} dx = \cos(1) - \int_1^\infty \frac{\cos(x)}{x^2} dx$. For $1 \leq x < \infty$, $\frac{-1}{x^2} \leq \frac{\cos(x)}{x^2} \leq \frac{1}{x^2}$, and since $\int_1^\infty \frac{dx}{x^2}$ converges, our integral converges.