Science One Math

Feb 4, 2019

Today

- Some more practice with trigonometric substitutions
- Partial fractions and other strategies to integrate rational functions

Last time: Trigonometric substitutions

When to use them?

- when the integrand has a square root
- when the argument of the radical can be reduced to either one of

$$1-\left(\frac{x}{a}\right)^2, \quad 1+\left(\frac{x}{a}\right)^2, \quad \left(\frac{x}{a}\right)^2-1$$

Let's do some more practice…

$$\int_{-\ln 2}^{0} e^x\sqrt{1-e^{2x}}\,dx$$

$\int_{-\ln 2}^{0} e^{x}\sqrt{1-e^{2x}}\,dx$ is equivalent to

A. $\int_{-\ln 2}^{0} u\sqrt{1-u^2}\,dx$
B. $\int_{-\ln 2}^{0} \sqrt{1-u^2}\,du$
C. $\int_{1/2}^{1} u\sqrt{1-u^2}\,du$
D. $\int_{1/2}^{1} \sqrt{1-u^2}\,du$
E. Either B or D

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B. $\int_{-\ln 2}^{0} \sqrt{1-u^2}\,du$

C. $\int_{1/2}^{1} u\sqrt{1-u^2}\,du$

D. $\boldsymbol{\int_{1/2}^{1} \sqrt{1-u^2}\,du}$

E. Either B or D

When you make a substitution, **remember to change the limits of integration!**

$\int_{-\ln 2}^{0} e^{x}\sqrt{1-e^{2x}}\,dx$ $\quad u = e^{x},\ du = e^{x}dx$ $\quad$ u-substitution

$\int_{1/2}^{1}\sqrt{1-u^2}\,du$ $\quad u = \sin\theta, du = \cos\theta d\theta$ $\quad$ trig substitution (inverse sub.)

$$\int_{\pi/6}^{\pi/2} cos^2\theta d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\theta=\pi/6}^{\theta=\pi/2} = \frac{1}{2}\left[\frac{\pi}{2} - \frac{\pi}{6} - \frac{1}{2}\frac{\sqrt{3}}{2}\right] = \frac{1}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

Note: Trig. substitutions are useful even if there is no root in the integrand.

$$\int \frac{dx}{(1+x^2)^2}$$

Trig. substitutions are useful even if there is no root in the integrand

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{sec^2\theta d\theta}{(1+tan^2\theta)^2} = \int \frac{sec^2\theta d\theta}{sec^4\theta} = \int cos^2\theta \, d\theta = \cdots$$

$x = \tan\theta$

Some practice (homework)

1) Find the area of the region enclosed by the ellipse $\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1$
2) Find the volume of the solid generated by revolving about the x-axis the region bounded by the curve $y = 4/(x^2 + 4)$, the x-axis, the lines $x = 0$ and $x = 2$.
3) $\int \frac{x^5}{\sqrt{x^2+2}} dx$
4) $\int \frac{1}{x\sqrt{5-x^2}} dx$

Which of the following is the **most efficient** strategy for computing $\int \frac{dx}{1+4x^2}$?

a) substitute $x = \frac{1}{2}\tan\theta$
b) substitute $x = 2\tan\theta$
c) substitute $x = 2\sec\theta$
d) substitute $x = \frac{1}{2}\cos\theta$
e) None of the above

Which of the following is the **most efficient** strategy for computing $\int \frac{dx}{1+4x^2}$?

a) $x = \frac{1}{2}\, tan\theta$ ☚ also correct but a little longer to compute

b) $x = 2\, tan\theta$

c) $x = 2\, sec\theta$

d) $x = \frac{1}{2} cos\theta$

e) None of the above

$u = 2x,\ du = 2dx$

$\int \frac{dx}{1+4x^2} = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}$ arctan(2x) + C ....don't forget the basics!

Another useful trick: Completing the square

…useful when we don't have a perfect square.

$\int \sqrt{x^2 - 2x + 2}\, dx$

$x^2 - 2x + 2 = (x^2 - 2x + 1) + 1 = (x-1)^2 + 1$, now trig. sub. $x - 1 = \tan\theta$

$\int \sqrt{2x - x^2}\, dx$

$-x^2 + 2x + 0 = (-x^2 + 2x - 1) + 1 = -(x-1)^2 + 1$, now trig. sub. $x - 1 = \sin\theta$

we'll use these tricks again later today…

OK, enough with trig substitutions....

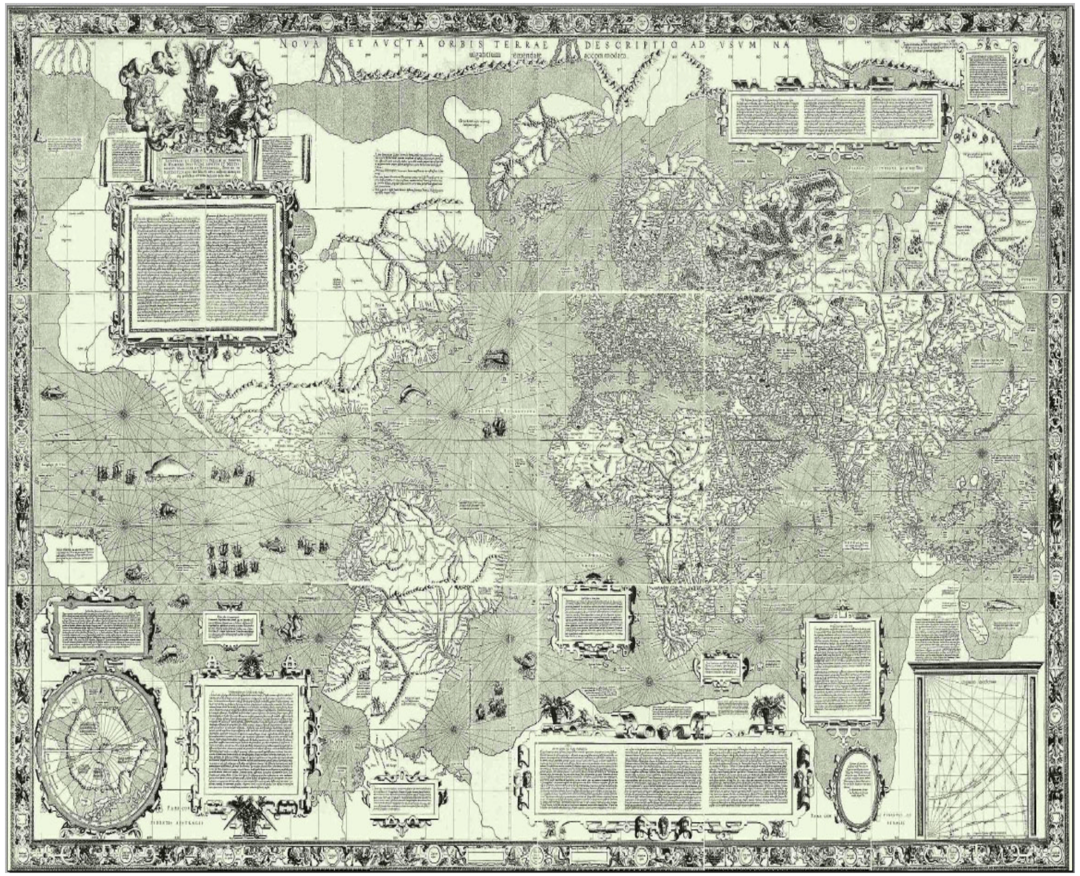
...back to trigonometric integrals. Last week we mentioned

$$\int \sec(x)\, dx = \ln|\sec x + \tan x| + C$$

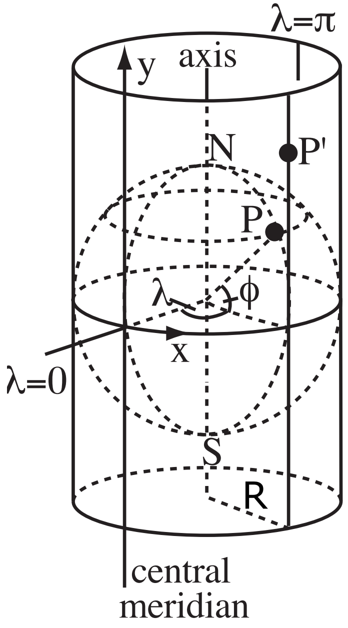
...famous problem in cartography...

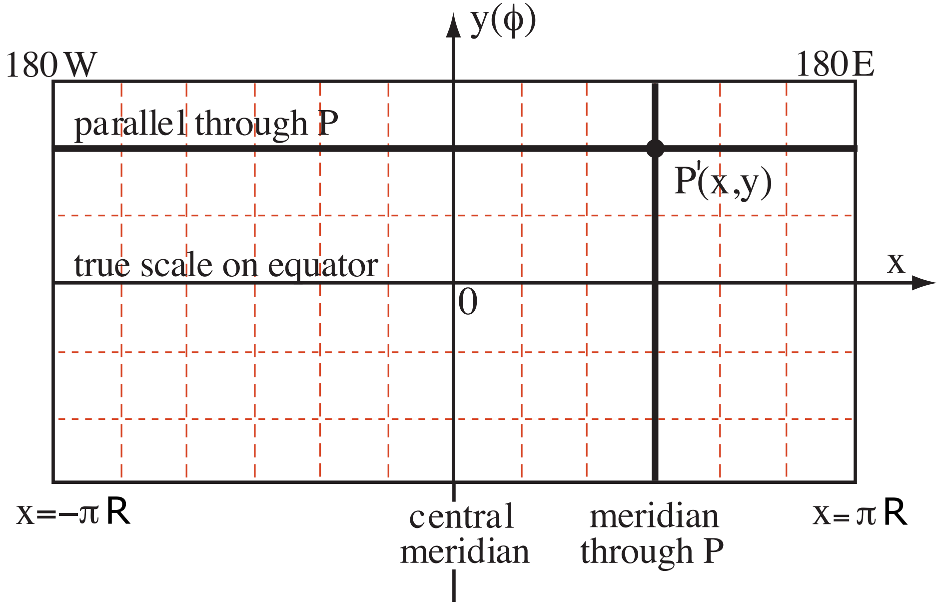
Mercator's map (1569)

"...In making this representation the world, we had to spread on a plane the surface of the sphere in such a way that the positions of places shall correspond on all sides with each other both in true direction and in distance. With this intention we had to employ a new proportion and a new arrangement of the meridians with reference to the parallels. For these reasons we have progressively increased the length of latitude towards each pole in proportion to the lengthening of parallels with reference to the equator."



On the sphere, the equation has length $2\pi R$ and the parallel at latitude ϕ has length $2\pi R\cos\phi$.





On the map, the equator and the parallel have **same** length. Thus the parallel has been stretched by a factor $\frac{1}{\cos\phi} = \sec\phi$. We want proportions and directions to be conserved (spherical square $\Leftrightarrow$ square on map).

Problem: Let $F(\phi)$ be distance of point P from equation on the map. So $F(0) = 0$. Find a formula for $F(\phi)$.

Proportions are conserved $\Longrightarrow \frac{R\Delta\phi}{2\pi R\cos\phi} = \frac{\Delta F}{2\pi} \Longrightarrow \frac{\Delta F}{\Delta\phi} = \frac{2\pi R}{2\pi R\cos\phi} = \sec\phi$

Hence $F(\phi) = \int_0^{\phi} \sec\varsigma \, d\varsigma$.

Some history of $\int \sec x \, dx$

(1645, first conjecture) $\int_0^\theta \sec t dt = \ln\left|\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right|$

(logarithms were invented in 1614 by Napier)

no, you don't need to remember these for the exam!

- (modern proof) $u = \sec x + \tan x,\ du = (\sec x\ \tan x + \sec^2 x)dx$

$$\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \int \frac{du}{u} = \ln|\sec x + \tan x| + C$$

- (one of the earliest proofs, 1670) $u = \sin x,\ du = \cos x \, dx$

$$\int \sec x \, dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{du}{1-u^2}$$ ...now what?

need strategies to integrate rational functions

- (a classic, sneaky substitution, mid 1700) $u = \tan\frac{x}{2}$, $\cos x = \frac{1-u^2}{1+u^2}$, $dx = \frac{2du}{1+u^2}$

$$\int \sec x \, dx = \int \frac{1}{\cos x} dx = \int \frac{1+u^2}{1-u^2} \frac{2du}{1+u^2} = 2\int \frac{1}{1-u^2} du$$ ...now what?

FYI: Tangent half-angle substitution

$u = \tan \frac{x}{2}$ (no, these won't be on the exam—no need to memorize them)

$$\cos x = \frac{1-u^2}{1+u^2}$$

$$\sin x = \frac{2u}{1+u^2}$$

$$\mathrm{d}x = \frac{2du}{1+u^2}$$

How to integrate $\int \frac{dx}{1-x^2}$?

We use a new technique based on the fact that we know how to integrate these

$$\int \frac{dx}{x-1} = \int \frac{du}{u} = \ln|x-1| + C$$
$$u = x - 1$$

$$\int \frac{dx}{x+1} = \int \frac{du}{u} = \ln|x+1| + C$$
$$u = x + 1$$

In general, we can integrate any rational functions of the form $\frac{1}{ax+b}$.

New technique: Partial Fractions

Start with an observation:

$$\frac{1/2}{x-1} - \frac{1/2}{x+1} = \frac{0.5(x+1)-0.5(x-1)}{(x-1)(x+1)} = \frac{1}{(x-1)(x+1)}$$

This means we can break the integrand up into "pieces" that are easy to integrate

$$\int \frac{1}{x^2-1}\,dx = \int \frac{1}{(x-1)(x+1)}\,dx = \int \frac{1/2}{x-1} - \frac{1/2}{x+1}\,dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

partial fractions ☛ easy to integrate

Other examples

$$\int \frac{1}{x^3-4x}\,dx = \int \frac{1}{x(x-2)(x+2)}\,dx$$

What does the integral above have in common with $\int \frac{dx}{1-x^2} = \int \frac{1}{(1-x)(1+x)}dx$?

What form is its partial fraction decomposition?

Other examples

$$\int \frac{1}{x^3-4x}\,dx = \int \frac{1}{x(x-2)(x+2)}\,dx$$

What does the integral above have in common with $\int \frac{dx}{1-x^2} = \int \frac{1}{(1-x)(1+x)}\,dx$?

What form is its partial fraction decomposition?

Common features: the numerator is a degree-zero poly, the denominator has only linear factors

Strategy: decompose the integrand into **partial fractions**, one fraction for each factor.

$\int \frac{1}{x^3-4x}dx = \int \frac{1}{x(x-2)(x+2)}dx$ seek decomposition of the form $\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}$

3 factors

(one fraction for each factor)

We want

$$\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}=\frac{1}{x(x-2)(x+2)}$$

Strategies for finding A, B, C:

<u>*Method 1*</u>: add partial fractions, choose coefficients so that numerators match (recall two polynomials are equal for all x if the like terms are equal)

<u>Method 2</u>: "cover-up" the factor $(x-a)$ and then substitute $x=a$. Repeat for each factor.

$$\Rightarrow\ A=-1/4, B=1/8, C=1/8$$

$$\int \frac{1}{x(x-2)(x+2)}dx = -\frac{1}{4}\int\frac{dx}{x}+\frac{1}{8}\int\frac{1}{x-2}dx+\frac{1}{8}\int\frac{1}{x+2}dx = -\frac{1}{4}\ln|x|+\frac{1}{8}\ln|x^2-4|+C$$

Partial Fractions: when to use

The **degree of numerator < degree of denominator** and

- the numerator is a degree-zero poly, the denominator has only linear factors

$$\int \frac{dx}{1-x^2}, \quad \int \frac{1}{x^3-4x}\,dx, \quad \int \frac{5}{2x^2-5x-3}\,dx, \quad \int \frac{1}{(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)}\,dx$$

- the numerator is a linear poly, the denominator has only linear factors

$$\int \frac{x+2}{x^3-x}\,dx \qquad \int \frac{5x+1}{(2x+1)(x-1)}\,dx$$

- the numerator is either a degree-zero or linear poly, the denominator has at least one (irreducible) quadratic factor

Partial Fractions: when to use

The **degree of numerator < degree of denominator** and

- the numerator is a degree-zero poly, the denominator has only linear factors

$$\int \frac{dx}{1-x^2}, \quad \int \frac{1}{x^3-4x}\,dx\,, \quad \int \frac{5}{2x^2-5x-3}\,dx\,, \quad \int \frac{1}{(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)}\,dx$$

- the numerator is a linear poly, the denominator has only linear factors

$$\int \frac{x+2}{x^3-x}\,dx\,, \quad \int \frac{5x+1}{(2x+1)(x-1)}\,dx\,, \quad \int \frac{cx+d}{(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)}\,dx$$

- the numerator is either a degree-zero or linear poly, the denominator has at least one (irreducible) quadratic factor

$$\int \frac{x+2}{x^3-x}\,dx \qquad x^3 - x = x(x+1)(x-1)$$

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} = \frac{x+2}{x^3-x}$$

$$A = -2,\ B = 1/2,\ C = 3/2$$

$$\int \frac{x+2}{x^3-x}\,dx = \int \frac{-2}{x} + \frac{1/2}{x+1} + \frac{3/2}{x-1}\,dx = -2\ln|x| + \frac{1}{2}\ln|x+1| + \frac{3}{2}\ln|x-1| + C.$$

$$\int \frac{5x+1}{(2x+1)(x-1)}dx$$

$$\frac{A}{(2x+1)} + \frac{B}{x-1} = \frac{5x+1}{(2x+1)(x-1)} \Rightarrow A = 1, B = 2$$

$$\int \frac{5x+1}{(2x+1)(x-1)}dx = \int \frac{dx}{2x+1} + \int \frac{2}{x-1}dx = \frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C$$

Partial Fractions: when to use

The **degree of numerator < degree of denominator** and

- the numerator is a degree-zero poly, the denominator has only linear factors

$$\int \frac{dx}{1-x^2}, \quad \int \frac{1}{x^3-4x}dx, \quad \int \frac{5}{2x^2-5x-3}dx, \quad \int \frac{1}{(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)}dx$$

- the numerator is a linear poly, the denominator has only linear factors

$$\int \frac{x+2}{x^3-x}dx, \quad \int \frac{5x+1}{(2x+1)(x-1)}dx, \quad \int \frac{cx+d}{(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)}dx$$

- the denominator has at least one (irreducible—cannot be factored) quadratic factor

$$\int \frac{2x^2+x+5}{(x^2+1)(x-2)}dx$$

Partial Fractions: when to use

The **degree of numerator < degree of denominator** and

- the numerator is a degree-zero poly, the denominator has only linear factors

$$\int \frac{dx}{1-x^2}, \quad \int \frac{1}{x^3-4x}dx, \quad \int \frac{5}{2x^2-5x-3}dx, \quad \int \frac{1}{(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)}dx$$

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$$\int \frac{x+2}{x^3-x}dx, \quad \int \frac{5x+1}{(2x+1)(x-1)}dx, \quad \int \frac{cx+d}{(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)}dx$$

- the denominator has at least one (irreducible—cannot be factored) quadratic factor

$$\int \frac{2x^2+x+5}{(x^2+1)(x-2)}dx$$

WARNING –in this case the partial fraction decomposition takes on a **different form**

$$\int \frac{2x^2+x+5}{(x^2+1)(x-2)}\,dx$$

$x^2 + 1$ is irreducible (can't be factored further)

Seek a decomposition of the form $\frac{Ax+B}{x^2+1} + \frac{C}{x-2}$

such that

$$\frac{Ax+B}{x^2+1} + \frac{C}{x-2} = \frac{2x^2+x+5}{(x^2+1)(x-2)}$$

$\Rightarrow$ A = −1, B = −1, C = 3

$$\int \frac{2x^2+x+5}{(x^2+1)(x-2)}\,dx = \int \frac{-x-1}{x^2+1} + \frac{3}{x-2}\,dx =$$

$$= -\int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx + \int \frac{3}{x-2}\,dx$$

$$= -\frac{1}{2}\ln(x^2+1) - \arctan(x) - 3\ln|x-2| + C$$

How to choose the form of partial fractions

1) When factors are **linear and distinct**

$$\mathrm{Q}(x) = (a_1x + b_1)(a_2x + b_2) \ldots (a_nx + b_n)$$

We seek a decomposition of the form

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

How to choose the form of partial fractions

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2) When at least on factor is quadratic (irreducible)

$$\mathrm{Q}(x) = (ax^2 + bx + c)(a_2x + b_2) \ldots (a_nx + b_n)$$

We seek a decomposition of the form

$$\frac{Ax+B}{ax^2+bx+c} + \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_n}{a_nx+b_n}$$

How to choose the form of partial fractions

1) When factors are **linear and distinct**

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \dots (a_nx + b_n)$$

We seek a decomposition of the form

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

2) When at least on factor is quadratic (irreducible)

$$Q(x) = (ax^2 + bx + c)(a_2x + b_2) \dots (a_nx + b_n)$$

We seek a decomposition of the form

$$\frac{Ax+B}{ax^2+bx+c} + \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_n}{a_nx+b_n}$$

Want partial fraction where the
degree of numerator = (degree of the denominator) -1

Which of the following is the most appropriate decomposition of

$$\frac{4x+5}{x\,(x-1)(x^2+1)}\ ?$$

$a)\ \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x^2+1}$

$b)\ \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$

$c)\ \frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+1}$

$d)\ \frac{A}{x}+\frac{B}{x-1}+\frac{Cx}{x^2+1}$

Which of the following is the most appropriate decomposition of

$$\frac{4x+5}{x\,(x-1)(x^2+1)}\ ?$$

a) $\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x^2+1}$

b) $\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$

c) $\boldsymbol{\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+1}}$

d) $\frac{A}{x}+\frac{B}{x-1}+\frac{Cx}{x^2+1}$

Partial Fractions: when to use

The **degree of numerator < degree of denominator** and

- the numerator is a degree-zero poly, the denominator has only linear factors
- the numerator is a linear poly, the denominator has only linear factors
- the denominator has at least one (irreducible—cannot be factored) quadratic factor

Exceptions:

1) when the degree of numerator = (degree of denominator) -1

$$\int \frac{x^2}{x^3-2}\,dx \qquad\qquad \int \frac{x^2-1}{x^3-4x}\,dx$$

Partial Fractions: when to use

The **degree of numerator < degree of denominator** and

- the numerator is a degree-zero poly, the denominator has only linear factors
- the numerator is a linear poly, the denominator has only linear factors
- the denominator has at least one (irreducible—cannot be factored) quadratic factor

Exceptions:

1) when the degree of numerator = (degree of denominator) -1

$$\int \frac{x^2}{x^3-2}\,dx = [u = x^3 - 2] = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|x^3-2| + C$$

$$\int \frac{x^2-1}{x^3-4x}\,dx \qquad u = x^3 - 4x, \quad du = (3x^2-4)dx$$

$$\frac{1}{3}\int \frac{3x^2-3}{x^3-4x}\,dx = \frac{1}{3}\int \frac{3x^2-3-1+1}{x^3-4x}\,dx = \frac{1}{3}\int \frac{3x^2-4}{x^3-2x}\,dx + \frac{1}{3}\int \frac{1}{x^3-4x}\,dx = \frac{1}{3}\int \frac{du}{u} + \text{[partial fractions]}$$

Partial Fractions: when to use

The **degree of numerator < degree of denominator** and

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- the numerator is a linear poly, the denominator has only linear factors
- the denominator has at least one (irreducible—cannot be factored) quadratic factor

Exceptions:

1) when the degree of numerator = (degree of denominator) -1

2) when the denominator is an irreducible quadratic polynomial

$$\int \frac{dx}{1+x^2} \qquad \int \frac{dx}{3x^2+4} \qquad \int \frac{dx}{x^2-2x+3}$$

Partial Fractions: when to use

The **degree of numerator < degree of denominator** and

- the numerator is a degree-zero poly, the denominator has only linear factors
- the numerator is a linear poly, the denominator has only linear factors
- the denominator has at least one (irreducible—cannot be factored) quadratic factor

Exceptions:

2) when the denominator is an irreducible quadratic polynomial

$\int \frac{dx}{1+x^2}$ = arctan(x) +C

$$\int \frac{dx}{3x^2+4} = \frac{1}{4}\int \frac{dx}{\left[\left(\frac{\sqrt{3}}{2}x\right)^2+1\right]} = \frac{1}{2\sqrt{3}}\arctan\left(\frac{\sqrt{3}}{2}x\right)+C$$

$$\int \frac{dx}{x^2-2x+3} = \int \frac{dx}{(x-1)^2+2} = \int \frac{1}{2}\frac{dx}{\left[(x-1)/\sqrt{2}\right]^2+1} = \frac{1}{2}\sqrt{2}\arctan\left(\frac{x-1}{\sqrt{2}}\right)+C$$

(complete the square)

Other examples of integrals of rational functions where partial fractions can't be used

- $\int \frac{2x+1}{x^2-2x+3} dx = \int \frac{2x-2+3}{x^2-2x+3} dx = \int \frac{2x-2}{x^2-2x+3} dx + 3 \int \frac{dx}{x^2-2x+1+2} = \cdots$

$u = x^2 - 2x + 3$ *complete the square*

- If quadratic factor is repeated: try trig. substitutions

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{sec^2\theta d\theta}{(1+tan^2\theta)^2} = \int \frac{sec^2\theta d\theta}{sec^4\theta} = \int cos^2\theta \, d\theta = \cdots$$

$x = \tan\theta$

Which of the following integrals would you solve using a partial fraction decomposition?

$i)\ \int \frac{2}{x^2+x}dx$ $\qquad ii\,)\ \int \frac{x}{x+2}dx$ $\qquad iii)\ \int \frac{dx}{x^2-2x+3}$

$iv)\ \int \frac{3x+2}{x^3-1}dx$ $\qquad v)\ \int \frac{3x^2-2}{x^3-2x-8}dx$

A. i) and iii)
B. i) and iv)
C. i) and iii) and iv)
D. iii) and iv) and v)
E. All of them

$$\int \frac{2}{x^2+x}\,dx = \int \frac{2}{x(x+1)}\,dx$$ ☚ need partial fractions

$$\int \frac{x}{x+2}\,dx = \int \frac{u-2}{u}\,du = \int \left(1-\frac{2}{u}\right)du = x+2\ -2\ln|x+2|+C$$

Alternatively, by long division $\int \frac{x}{x+2}\,dx = \int 1-\frac{2}{x+2}\,dx = x-2\ln|x+2|+C$

$$\int \frac{dx}{x^2-2x+3} = \int \frac{dx}{(x-1)^2+2} = \frac{1}{2}\int \frac{dx}{\left(\frac{x-1}{\sqrt{2}}\right)^2+1} = \frac{1}{2}\int \frac{\sqrt{2}\,du}{u^2+1} = \frac{\sqrt{2}}{2}\arctan\left(\frac{x-1}{\sqrt{2}}\right)+C$$

$$\int \frac{3x+2}{x^3-1}\,dx = \int \frac{3x+2}{(x-1)(x^2+x+1)}\,dx$$ ☚ need partial fractions

$$\int \frac{3x^2-2}{x^3-2x-8}\,dx = \int \frac{du}{u} = \ln|x^3-2x-8|+C$$

Different types of $\int \frac{P(x)}{Q(x)} dx$

- When degree(P) < degree(Q) ☞ partial fractions
  (with a few noticeable exceptions)

- When degree(P)≥ degree(Q) ☞ divide polynomials

Different types of $\int \frac{P(x)}{Q(x)} dx$

- When degree(P) < degree(Q) ☞ partial fractions

  (with a few noticeable exceptions)

- When degree(P)≥ degree(Q) ☞ divide polynomials

  $\int \frac{x^3+x}{x-1} dx$

$= \int x^2 + x + 2 + \frac{2}{x-1} dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + 2\ln|x-1| + C$

Recap: How to compute $\int \frac{P(x)}{Q(x)} dx$

- **If degree of P ≥ degree of Q**
  - Divide first
- **If degree of P = (degree of Q) -1**
  - Check if $u = Q(x)$ works
- **If degree of P < (degree of Q) -1**
  - Factor Q(x)
    - If Q(x) = (linear factors)(quadratic factors)
      - Partial fraction decomposition
    - If Q(x) = (irreducible) quadratic poly
      - try completing the square
      - try u-sub reducing the integrand to $1/(1+u^2)$

Summary of Integration techniques we learned

- **u-substitution**: works for a variety of integrands
- **by parts**: ideal to integrate
  - exponential or log functions (usually multiplied by a poly)
  - inverse trig. functions (usually multiplied by a poly)
  - some product of sines and cosines (or tangents and secants)
- **trigonometric substitution**: works for integrands with roots (and sometimes without roots)
- **partial fractions**: for rational integrands
- **completing the square**: also for rational integrals (and sometimes for integrands with root)

$$\int \frac{\sin^3 x}{\cos x}\, dx$$

$$\int r^4 \ln r \, dr$$

$$\int \frac{x}{x^4+x^2+1}\, dx$$

$$\int e^{x+e^x}\, dx$$

$$\int \frac{\arctan x}{x^2}\, dx$$

$$\int \frac{\sin(2x)}{1+\cos^4 x}\, dx$$

$$\int \frac{dx}{\sqrt{x}\left(2+\sqrt{x}\right)^4}$$

$$\int \sin^3 x \, dx$$