Improper Integrals

What is the area under the graph $y = e^{-x}$ for $0 \leq x < \infty$?

For any $R > 0$, the area under the graph for $0 \leq x \leq R$ is

$$\int_0^R e^{-x} dx = -e^{-x}\Big|_0^R = 1 - e^{-R}$$

To define (and compute) the area over the infinite interval $0 \leq x < \infty$, take $R \to \infty$:

$$\int_0^\infty e^{-x} dx = \lim_{R\to\infty} \int_0^R e^{-x} dx = \lim_{R\to\infty} (1 - e^{-R}) = \boxed{1}$$

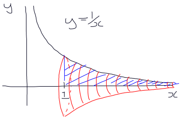
Improper Integrals (Type I: infinite integration domain)

$\int_a^\infty f(x)dx = \lim_{R\to\infty} \int_a^R f(x)dx$ if this limit exists

$\int_{-\infty}^b f(x)dx = \lim_{r\to-\infty} \int_r^b f(x)dx$ if this limit exists

If the limit exists, the improper integral is said to **converge**.
If not, it is said to **diverge**.

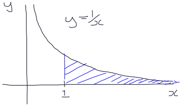
How much ice cream is needed to fill an infinite ice cream cone obtained by rotating $y = \frac{1}{x}$, $1 \leq x < \infty$, about the x-axis?



$$V = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \pi \lim_{R\to\infty} \int_1^R \frac{1}{x^2} dx = \pi \lim_{R\to\infty} \left(-\frac{1}{x}\right) \Big|_1^R$$
$$= \pi \lim_{R\to\infty} \left(-\frac{1}{R} + 1\right) = \boxed{\pi}$$ (this improper integral **converges**)

Now suppose you have filled the infinite cone. Slice it along the xy-plane. What is the area of this cross-section?

It is twice this area:



$$A = 2\int_1^\infty \frac{1}{x} dx = 2 \lim_{R\to\infty} \int_1^R \frac{1}{x} dx = 2 \lim_{R\to\infty} \ln(x)|_1^R$$
$$= 2 \lim_{R\to\infty} (\ln(R) - \ln(1)) = 2 \lim_{R\to\infty} \ln(R) = \boxed{\infty}$$ (**diverges** !)

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{x}{1+x^2}\,dx + \int_{0}^{\infty} \frac{x}{1+x^2}\,dx$$

$$= \lim_{r\to-\infty} \int_{r}^{0} \frac{x}{1+x^2}\,dx + \lim_{R\to\infty} \int_{0}^{R} \frac{x}{1+x^2}\,dx$$

$$= \lim_{r\to-\infty} \frac{1}{2}\ln(1+x^2)\Big|_{r}^{0} + \lim_{R\to\infty} \frac{1}{2}\ln(1+x^2)\Big|_{0}^{R}$$

$$= \lim_{r\to-\infty} \frac{1}{2}\left(-\ln(1+r^2)\right) + \lim_{R\to\infty} \frac{1}{2}\left(\ln(1+R^2)\right)$$

These limits do not exist! So this improper integral **diverges**.

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to-\infty} \int_{r}^{c} f(x)dx + \lim_{R\to\infty} \int_{c}^{R} f(x)dx$$

converges only if both limits exist.

Warning: $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{x}{1+x^2}\,dx = \lim_{R\to\infty} \frac{1}{2}\ln(1+x^2)\Big|_{-R}^{R}$

$$= \lim_{R\to\infty} \frac{1}{2}\left(\ln(1+R^2) - \ln(1+R^2)\right) = \lim_{R\to\infty} 0 = 0$$

is **wrong**!

For $\int_a^\infty f(x)dx$ to converge, $f(x)$ must go to zero as $x \to \infty$, fast enough: for example, evaluate the "p-integral"

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{R\to\infty} \left\{ \begin{array}{ll} \frac{x^{1-p}}{1-p}\Big|_1^R = \frac{R^{1-p}-1}{1-p} & p \neq 1 \\ \ln(x)\Big|_1^R = \ln(R) & p = 1 \end{array} \right\}$$

$$\left\{ \begin{array}{ll} < \infty & p > 1 \\ \infty & p \leq 1 \end{array} \right.$$

Notice that $\int_1^\infty \frac{1}{x}dx$ diverges, but this power $(p = 1)$ is the "borderline" case between convergent and divergent behaviour.