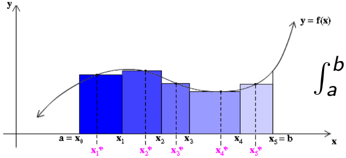
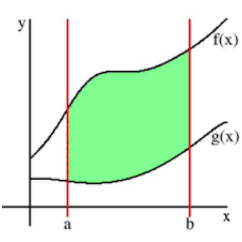
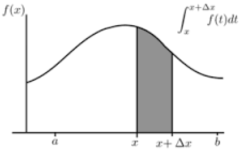
some things we now know about $\int$ ...



$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^n f(x_k^*)\Delta x$$



$$A = \int_a^b (f(x) - g(x))\, dx$$



FTC

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$
$$\int_a^b f(x)dx = F(b) - F(a), \quad F' = f$$

Which functions do we already know how to integrate (antidifferentiate) explicitly?

- powers: $\int t^n \, dt = \frac{1}{n+1} t^{n+1} + C \quad (n \neq -1)$
  $\int \frac{1}{t} \, dt = \ln|t| + C$

- polynomials: $\int \left(a_0 + a_1 y + a_2 y^2 + \cdots + a_n y^n\right) \, dy$
  $= a_0 y + \frac{1}{2} a_1 y^2 + \frac{1}{3} a_2 y^3 + \cdots + \frac{1}{n+1} a_n y^{n+1} + C$

- exponential: $\int e^u \, du = e^u + C, \qquad \int e^{-u} \, du = -e^{-u} + C$

- some trig: $\int \cos(x) \, dx = \sin(x) + C$
  $\int \sin(x) \, dx = -\cos(x) + C$

Next task: master some tricks to integrate more complicated functions...

Substitution Rule

- recall the chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$
- so the chain rule "in reverse" says:

$\int f'(g(x))g'(x)dx = f(g(x)) + C$	**Substitution Rule**

- we get the same thing by **substitution** into the integral of
  $u = g(x), \quad \frac{du}{dx} = g'(x) \to du = g'(x)dx$:

$$\int f'(g(x))g'(x)dx = \int f'(u)du = f(u) + C = f(g(x)) + C.$$

Example: $\int x^2 \sin(x^3) dx$

$u = x^3, \quad du = \frac{du}{dx}dx = 3x^2 dx \implies x^2 dx = \frac{1}{3}du$

$\int x^2 \sin(x^3)dx = \frac{1}{3}\int \sin(u)du = -\frac{1}{3}\cos(u) + C = -\frac{1}{3}\cos(x^3) + C$

- it's easy to check if you got an antiderivative right, just by differentiating – so do it!

Substitution Rule for Definite Integrals

Example: $\int_1^e \frac{dx}{x(1+(\ln(x))^2)} = ?$

<u>Method 1</u>. Find an antiderivative, by substitution:

$$u = \ln(x) \implies du = \frac{d\ln(x)}{dx}dx = \frac{1}{x}dx$$

so $\int \frac{dx}{x(1+(\ln(x))^2)} = \int \frac{du}{1+u^2} = \tan^{-1}(u) + C = \tan^{-1}(ln(x)) + C$

and use the FTC:

$$\int_1^e \frac{dx}{x(1+(\ln(x))^2)} = \tan^{-1}(\ln(x))|_1^e = \tan^{-1}(\ln(e)) - \tan^{-1}(\ln(1))$$

$$= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$$

<u>Method 2</u>. Keep the limits, substituting in them as well:

$$\int_1^e \frac{dx}{x(1+(\ln(x))^2)} = \int_{u=\ln(1)}^{u=\ln(e)} \frac{du}{1+u^2} = \int_0^1 \frac{du}{1+u^2} = \tan^{-1}(u)|_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0) = \boxed{\frac{\pi}{4}}$$

Some examples to try:

1. $\int \left(5e^{-3x} + \sin(2x)\right)\ dx =$
2. $\int \frac{s}{s+1}\ ds =$
3. $\int_{4\pi^2}^{9\pi^2} \frac{\sin(\sqrt{t})}{\sqrt{t}}\ dt =$
4. $\int_0^1 x\sqrt{1-x}\ dx =$
5. $\int_0^1 \frac{e^{2y}}{e^y+1}\ dy =$
6. $\int \frac{dt}{t^2+4t+5} =$
7. $\int \frac{t}{t^2+4t+5}\ dt =$

Some more:

1 solve $\begin{cases} \frac{dP}{dt} = \frac{e^{-P^2}}{P} \\ P(0) = 1 \end{cases}$

- $\frac{dP}{dt} = \frac{e^{-P^2}}{P} \implies e^{P^2} P dP = dt \implies \int e^{P^2} P dP = \int dt = t + C$
- sub $u = P^2 \implies du = 2PdP$ to find

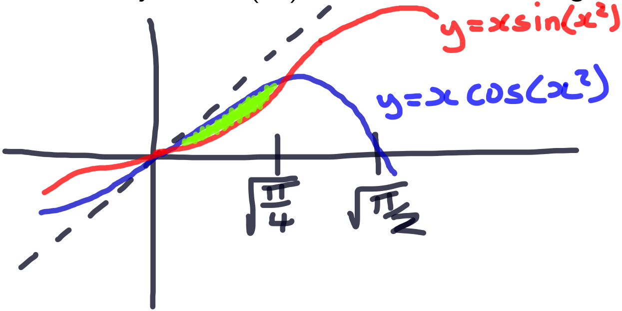
$$\int e^{P^2} P dP = \frac{1}{2} \int e^u du = \frac{1}{2} e^u = \frac{1}{2} e^{P^2}$$

(ignore the constant since we already have one on the other side)

- so $\frac{1}{2} e^{P^2} = t + C$, and $P(0) = 1 \implies \frac{1}{2} e = C$

so $e^{P^2} = 2t + e \implies P^2 = \ln(2t + e) \implies \boxed{P(t) = \sqrt{\ln(2t + e)}}$.

2 Find the area of the piece of the region below $y = x\cos(x^2)$ and above $y = x\sin(x^2)$ which touches the origin.



- $A = \int_0^{\sqrt{\frac{\pi}{4}}} \left(x\cos(x^2) - x\sin(x^2)\right) dx$
- sub $u = x^2$, so $du = 2xdx \implies xdx = \frac{1}{2}du$
- so $A = \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos(u) - \sin(u))\, du = \frac{1}{2}\left(\sin(u) + \cos(u)\right)\big|_0^{\frac{\pi}{4}}$
  $= \frac{1}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) = \boxed{\frac{1}{2}(\sqrt{2} - 1)}.$