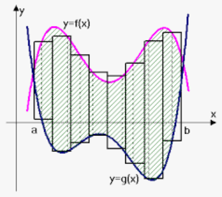
Computing volumes with integrals

To compute the area of a region, 

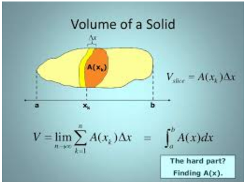
slice it,

and approximate slices by rectangles: $A = \int_a^b (f(x) - g(x))\,dx$.

To compute the volume of a solid 

slice it, and

approximate the slices by "cylinders":



$$V = \int_a^b A(x)dx$$
$$A(x) = \text{ cross-sectional area at } x$$

warm-up example: find the volume of a ball of radius R

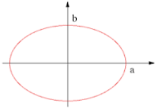
- arranging the ball so that one diameter is the interval $[-R,\ R]$ on the x-axis, the cross-section at x is a disk of radius $\sqrt{R^2 - x^2}$, whose area is $A(x) = \pi(R^2 - x^2)$, so

$$V = \int_{-R}^{R} A(x)dx = \int_{-R}^{R} \pi(R^2 - x^2)dx = \pi \left(R^2 x - \frac{1}{3}x^3 \right) \Big|_{-R}^{R}$$
$$= \pi \left(R^3 - \frac{1}{3}R^3 - R^2(-R) - \frac{1}{3}(-R)^3 \right) = \frac{4}{3}\pi R^3$$

Example: Find the volume of the tetrahedron (triangular pyramid) with base in the xy-plane bounded by the axes and $x + y = a$ ($a > 0$), and with vertex h units above the origin.

Draw a picture! The cross-section at height z above the xy-plane is a an isoceles right triangle with side lengths (of the equal sides) $\frac{a}{h}(h - z)$ (by similar triangles), hence of area $A(z) = \frac{1}{2}\frac{a^2}{h^2}(h - z)^2$. So its volume is

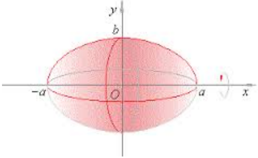
$$V = \int_0^h A(z)dz = \frac{a^2}{2h^2}\int_0^h (h-z)^2 dz = \frac{a^2}{2h^2}(-\frac{1}{3}(h-z)^3)|_0^h = \frac{1}{6}a^2h.$$



Example: consider $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: – an **ellipse**.

Find the volume of the solid swept out by rotating the region

inside this ellipse about the x-axis:



- the cross-section at x is a disk of radius $b\sqrt{1 - \frac{x^2}{a^2}}$, whose area is $A(x) = \pi b^2(1 - \frac{x^2}{a^2})$, so

$$V = 2\int_0^a A(x)dx = 2\int_0^a \pi b^2(1 - \frac{x^2}{a^2})dx = 2\pi b^2\left(x - \frac{x^3}{3a^2}\right)\Big|_0^a$$

$$= 2\pi b^2\left(a - \frac{a}{3}\right) = \frac{4}{3}\pi a b^2$$

This is sometimes called the **“disk method”** for computing the volume of a **“solid of revolution”**.

Example: Find the volume of the solid produced when the triangle bounded by $x = 0$, $y = 1$ and $y = x$ is rotated around the x-axis.

- cross-section at x is a "washer" of outer radius 1 and inner radius x, whose area is $A(x) = \pi(1)^2 - \pi(x)^2 = \pi(1 - x^2)$, so

$$V = \int_0^1 \pi(1 - x^2)dx = \pi(x - \frac{1}{3}x^3)|_0^1 = \frac{2}{3}\pi$$ ✓

This variant of the "disk method" is sometimes called – surprise – the **"washer method"**.

Example: Find the volume of the solid produced when the region $\{0 \le x \le \sqrt{\pi},\ 0 \le y \le \sin(x^2)\}$ is rotated around the y-axis.

- a slice of the original region between x and $x + \Delta x$ is nearly a rectangle of height $\sin(x^2)$ and width Δx, hence area $\sin(x^2)\Delta x$.
- when this (near) rectangle is rotated about the y-axis, it produces a "cylindrical shell" of radius $\approx x$, and so of volume $\approx (2\pi x)(\sin(x^2)\Delta x)$.
- adding these all up and taking $\Delta x \to 0$:

$$V = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2) dx \qquad \left(u = x^2 \implies du = 2x dx \right)$$

$$= \pi \int_0^{\pi} \sin(u) du = -\pi \cos(u)|_0^{\pi} = -\pi(-1-1) = 2\pi.$$

This is sometimes called the **"(cylindrical) shell method"** for computing the volume of a solid of revolution.

Example: the region bounded by $y = x$ and $y = \sqrt{x}$ is rotated about the y-axis. Find the volume by two different methods.