Science One Integral Calculus

January 2018

Happy New Year!

Differential Calculus central idea: The derivative

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☞ (geometrically) **slope of tangent** line to graph of f at x

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What is the mathematical definition of $f'(x)$? It's a **limit**!

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ or equivalently $$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

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- ☞ (geometrically) **area** of region under f curve above $[a, b]$
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Our main goals this term…

- Give a **precise definition** of the definite integral
- Find a **fundamental connection** with the derivative (The Fundamental Theorem of Calculus)
- Master **integration techniques** to compute complicated antiderivatives
- Apply integration to a variety of **science contexts**

➤**Today's goal:** Give a **mathematical definition** of the definite integral

The area problem:
Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

What is *area*?

Easy for regions with straight sides...not so easy for regions with curved sides!

We need a precise definition of area.

Example: Find the area under $f(x) = x^2$ on $[0, 1]$.

- Worksheet

We found that the sum S_n of areas of n rectangles converges as $n \rightarrow \infty$

We define area as a limit, $S = lim_{n \rightarrow \infty} S_n$

The definite integral

Consider the region under the curve $y = f(x)$ above $[a, b]$.

- Take n vertical strip of equal width $\Delta x = (b - a)/n$

  n intervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots [x_{i-1}, x_i], \ldots [x_{n-1}, x_n]$.

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  i-th strip has area $\Delta x\, f(x_i^*)$
- Sum the areas of all rectangles

  $$S_n = \Delta x\, f(x_1^*) + \Delta x\, f(x_2^*) + \cdots + \Delta x\, f(x_i^*) + \cdots + \Delta x\, f(x_n^*)$$

  where sample point x_i^* is *any* number in the interval $[x_{i-1}, x_i]$.

Sigma Notation

Convenient notation for writing long sums.

E.g. the sum of the first 10 squares can be written as

$$1 + 2^2 + 3^2 + \cdots + 10^2 = \sum_{k=1}^{10} k^2$$

This reads as

" the sum from k equals 1 to 10 of k^2 "

k is called **summation index (dummy variable).**

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- Take the limit of the sum for $n \to \infty$

  $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$ this is the desired area (if the limit exists)

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Riemann Sum