Science One Math

Jan 21 2019

Today’s Goal: Compute Trigonometric Integrals

- Apply the *technique of substitution* to compute **trigonometric integrals** of the form

$\int \sin^n(x) \cos^m(x) dx$ for both $m > 1$ and $n > 1$

and some cases of the form

$\int \tan^m(x) \sec^n(x) dx$.

Do we know how to compute these integrals?

$\int \sec^2(x)dx$

$\int \cos(x)\sin(x)dx$

$\int \sin^2(x)\cos^3(x)dx$

$\int \sin^3(x)dx$

$\int \tan^3(x)\sec^5(x)dx$

Can we compute the following integrals?

✓ $\int \sec^2(x)dx$ [by inspection] $\Rightarrow$ = $\tan(x)$ + C

$\int \cos(x)\sin(x)dx$

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[sub: u=sin(x)] $\Rightarrow$ = $\int u\, du = \frac{\sin^2(x)}{2}$ + D same answer?

Recall $\sin^2(x) = 1 - \cos^2(x)$

$\cos^2(x) = 1 - \sin^2(x)$ trig. identities will come handy!

$-\frac{\cos^2(x)}{2}$ + C is equivalent to $\frac{\sin^2(x)}{2}$ + D .

Can we compute the following integrals?

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✓ $\int \cos(x)\sin(x)dx$ [sub: u=cos(x)] $\Rightarrow$ = $\int -u\,du = -\frac{cos^2(x)}{2}$ + C

$\int \sin^2(x)\cos^3(x)dx$ = ???

$\int \sin^3(x)dx$ = ???

$\int \tan^3(x)\sec^5(x)dx$ = ???

Useful stuff….

Basic antiderivatives:

$\int \sin(x)dx = -\cos(x) + C$

$\int \cos(x)dx = \sin(x) + C$

$\int \sec^2(x)dx = \tan(x) + C$

$\int \tan(x)\sec(x)\,dx = \sec(x) + C$

Basic trig identity

$\sin^2(x) + \cos^2(x) = 1$

$\tan^2(x) + 1 = \sec^2(x)$

Double-angle formulae

$\sin^2(x) = \frac{1}{2}\,[1 - \cos(2x)]$

$\cos^2(x) = \frac{1}{2}\,[1 + \cos(2x)]$

Sometimes it is also useful to know $\sin(2x) = 2\sin(x)\cos(x)$

Simple cases

1) $\int \sin(x)\cos^2(x)dx$
2) $\int \sin^3(x)\cos(x)dx$
3) $\int \sin^2(x)dx$
4) $\int \sin^3(x)dx$

Simple cases: Products

1) $\int \sin(x)\cos^2(x)dx$ [sub: u=cos(x)] = $\int -u^2\, du = -\frac{\cos^3(x)}{3} + C$

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2) $\int \sin^3(x)\cos(x)\,dx$

A. $u = \cos(x)$

B. $u = \sin(x)$

C. neither

Simple cases: Products

1) $\int \sin(x)\cos^2(x)dx$ [sub: u=cos(x)] = $\int -u^2\,du = -\frac{\cos^3(x)}{3} + C$

2) $\int \sin^3(x)\cos(x)dx$ [sub: u=sin(x)] = $\int u^3\,du = \frac{\sin^4(x)}{4} + C$

Simple cases: Products

1) $\int \mathbf{sin}(\boldsymbol{x})\,\mathbf{cos}^{2}(\boldsymbol{x})dx$ [sub: u=cos(x)]

2) $\int \mathbf{sin}^{3}(\boldsymbol{x})\,\mathbf{cos}(\boldsymbol{x})dx$ [sub: u=sin(x)]

Product of
one sine (or cosine) and
some **powers of cosine (or sine)**

Simple cases: Products

1) $\int \sin(x)\cos^2(x)dx$ [sub: u=cos(x)]

2) $\int \sin^3(x)\cos(x)dx$ [sub: u=sin(x)]

Product of **one sine (or cosine)** and some **powers of cosine (or sine)**

General case:

For any n

$$\int \sin^n(x)\cos(x)dx$$

$$u = \sin(x)$$

$$du = \cos(x)dx$$

For any m

$$\int \cos^m(x)\sin(x)dx$$

$$u = \cos(x)$$

$$du = -\sin(x)dx$$

Simple cases: Single Powers

3) $\int \sin^2(x)dx$

4) $\int \sin^3(x)dx$

3) $\int \sin^2(x)dx$ [use $\sin^2(x) = \frac{1}{2}\,[1 - \cos(2x)]$]

$$= \int \frac{1}{2}\,[1 - \cos(2x)]\,dx = \frac{1}{2}\,[x - \frac{1}{2}\sin(2x)] + C$$

3) $\int \sin^2(x)dx$ [use $\sin^2(x) = \frac{1}{2}[1 - \cos(2x)]$]

$$= \int \frac{1}{2}[1 - \cos(2x)]\, dx = \frac{1}{2}[x - \frac{1}{2}\sin(2x)] + C$$

4) $\int \sin^3(x)dx = \int \sin^2(x)\sin(x)\, dx =$ [use $\sin^2(x) = 1 - \cos^2(x)$]

$$= \int (1 - \cos^2(x))\sin(x)\, dx = \int \sin(x)dx - \int \cos^2(x)\sin(x)dx =$$

$$= -\cos(x) - \int -u^2 du = -\cos(x) + \frac{\cos^3(x)}{3} + C$$

Simple cases: Single Powers

3) $\int \sin^{2}(x)dx$ [double-angle formula] — powers of sine — even power

4) $\int \sin^{3}(x)dx$ [trig. identity + u-sub] — no cosine term — odd power

(same strategy works for powers of cosine, no sine term)

Simple cases: Single Powers

3) $\int \sin^{2}(x)dx$ [double-angle formula] — powers of sine, even power

4) $\int \sin^{3}(x)dx$ [trig. identity + u-sub] — no cosine term, odd power

General case

- For **even** $n > 1$,

$\int \sin^{n}(x)dx$ $\sin^2(x) = \frac{1}{2}[1 - \cos(2x)]$

$\int \cos^{n}(x)dx$ $\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$

- For **odd** $n > 1$

$\int \sin^{n}(x)dx$ isolate $\sin(x)$, convert the rest to powers of cosine, $u = \cos(x)$

$$\sin^2(x) = 1 - \cos^2(x)$$

$\int \cos^{n}(x)dx$ isolate cos(x), convert the rest to powers of sine, u=sin(x)

$$\cos^2(x) = 1 - \sin^2(x)$$

Simple cases: Single Powers

3) $\int \sin^{2}(x)dx$ [double-angle formula]
4) $\int \sin^{3}(x)dx$ [trig. Identity + u-sub]

powers of sine / no cosine term — even power / odd power

General case

- For **even** $n > 1$,

$\int \sin^{n}(x)dx$ $\sin^{2}(x) = \frac{1}{2}[1 - \cos(2x)]$

$\int \cos^{n}(x)dx$ $\cos^{2}(x) = \frac{1}{2}[1 + \cos(2x)]$

alternatively:
Integration by parts
(on Wednesday)

- For **odd** $n > 1$

$\int \sin^{n}(x)dx$ isolate sin(x), convert the rest to powers of cosine, u=cos(x)

$$\sin^{2}(x) = 1 - \cos^{2}(x)$$

$\int \cos^{n}(x)dx$ isolate cos(x), convert the rest to powers of sine, u=sin(x)

$$\cos^{2}(x) = 1 - \sin^{2}(x)$$

other examples...

$\int \sin^2(x)\cos^2(x)dx$ [this is neither $\int \sin^n(x)\cos(x)dx$ nor $\int \cos^n(x)\sin(x)dx$]

$$\int \sin^2(x)\cos^2(x)dx$$

Strategy: use trig identity, convert integrand to powers of sine or cosine only

$$\int (1-\cos^2(x))\cos^2(x)dx$$

$$\int \sin^2(x)\cos^2(x)dx$$

Strategy: use trig identity, convert integrand to powers of sine or cosine only

$$\int (1-\cos^2(x))\cos^2(x)dx = \int [\cos^2(x) - \cos^4(x)]dx = \int \cos^{\mathbf{2}}(x)dx - \int \cos^{\mathbf{4}}(x)dx$$

single powers of cosine $\Rightarrow$ use double-angle formulae for cosine

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Strategy: use trig identity, convert integrand to powers of sine or cosine only

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single powers of cosine $\Rightarrow$ use double-angle formulae for cosine

$$\int \cos^{\mathbf{2}}(x)dx = \int \tfrac{1}{2}[1+\cos(2x)]\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$

$$\int \cos^4(x)dx = \int \tfrac{1}{4}[1+\cos(2x)]^2 dx = \frac{1}{4}\int [1+2\cos(2x)+\cos^2(2x)]dx =$$

$$= \frac{1}{4}\left(x + \sin(2x) + \int \cos^2(2x)dx\right) =$$

$$= \frac{1}{4}\left(x + \sin(2x) + \int \tfrac{1}{2}[1+\cos(4x)]dx\right) =$$

$$= \frac{1}{4}\left(x + \sin(2x) + \frac{1}{2}x + \frac{1}{8}\sin(4x)\right) + C$$

doable, but time consuming!

alternative method on Wednesday

other examples…

$\int \sin^3(x)\cos^2(x)dx =$ [Strategy: reduce the **odd** power]

$= \int \sin^2(x)\cos^2(x)\sin(x)\,dx$

$\int \sin^3(x)\cos^2(x)dx =$ [Strategy: reduce the odd power]

$= \int \sin^2(x) \cos^2(x) \sin(x)\, dx$

[Convert to powers of cosines, then substitute $u = \cos(x)$]

$= \int (1 - \cos^2(x))\cos^2(x) \sin(x)\, dx =$

$= \int \cos^2(x) \sin(x)\, dx - \int \cos^4(x) \sin(x)\, dx = \int -u^2 du + \int u^4\, du = \cdots$

$\int \sin^3(x)\cos^2(x)dx =$ [Strategy: reduce the odd power]

$= \int \sin^2(x) \cos^2(x) \sin(x)\, dx$

[Convert to powers of cosines, then substitute $u = \cos(x)$]

$= \int (1 - \cos^2(x))\cos^2(x) \sin(x)\, dx =$

$= \int \cos^2(x) \sin(x)\, dx - \int \cos^4(x) \sin(x)\, dx = \int -u^2 du + \int u^4\, du = \cdots$

Alternatively, $\int \sin^3(x)\cos^2(x)dx = \int \sin^2(x) \cos^2(x) \sin(x)\, dx$

[convert to powers of sine, compute integrals of powers of sine]

$= \int \sin^2(x)(1 - \sin^2(x))\sin(x)\, dx =$

$= \int \sin^3(x)dx - \int \sin^5(x)dx$ **odd** powers of sine $\Rightarrow$ reduce power,

doable, longer!

convert to cosine,

u-substitution...

Practice

$\int \cos^3(x) \sin^2(x) dx$

The **most efficient** strategy to compute the above integral above is to convert it to which of the following integrals?

A. $\int \cos^2(x) \sin^2(x) \cos(x)\, dx$

B. $\int \cos^3(x) \sin(x) \sin(x)\, dx$

C. $\int \cos^3(x) \, (1 - \cos^2(x)) dx$

$\int \cos^3(x)\sin^2(x)dx = \int \cos^2(x)\sin^2(x)\cos(x)\,dx$ =

= $\int (1-\sin^2(x))\sin^2(x)\cos(x)\,dx$ =

$= \int \sin^2(x)\cos(x)\,dx - \int \sin^4(x)\cos(x)\,dx$ =

$= \int u^2 du - \int u^4\,du = \cdots$

$\int \cos^3(x)\sin^2(x)dx = \int \cos^3(x)\,(1-\cos^2(x))dx$

$\int \cos^3(x)dx = \int \cos^2(x)\cos(x)\,dx = \int (1-\sin^2(x))\cos(x)\,dx =$

$= \int \cos(x)\,dx - \int u^2 du = \cdots$

$\int \cos^5(x)dx = \int \cos^4(x)\cos(x)\,dx = \int (1-\sin^2(x))^2\cos(x)\,dx =$

$= \int (1-2\sin^2(x)+\sin^4(x))\cos(x)\,dx = \int (1-2u^2+u^4)du = \cdots$

much longer…

Recap

A) $\int \sin^n(x)\cos(x)dx$ or $\int \cos^n(x)\sin(x)dx$

↳ use u-substitution

Recap

A) $\int \sin^n(x)\cos(x)dx$ or $\int \cos^n(x)\sin(x)dx$

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B) $\int \sin^n(x)dx$ or $\int \cos^n(x)dx$

if n is **even** use double-angle formulae

if n is **odd** reduce the power, use trig identify

Recap

A) $\int \sin^n(x)\cos(x)dx$ or $\int \cos^n(x)\sin(x)dx$

↳ use u-substitution

B) $\int \sin^n(x)dx$ or $\int \cos^n(x)dx$

↳ if n is **even** use double-angle formulae

if n is **odd** reduce the power, use trig identify

C) $\int \sin^n(x)\cos^m(x)dx$ for both $m > 1$ and $n > 1$

↳ if one power is **odd**, reduce the odd power, use trig identity, reduce to A)

if both powers are **even**, use trig identify, reduce to B)

Recap: Every $\sin^n(x)\cos^m(x)$ can be integrated!

A) $\int \sin^n(x)\cos(x)dx$ or $\int \cos^n(x)\sin(x)dx$

↳ use u-substitution

B) $\int \sin^n(x)dx$ or $\int \cos^n(x)dx$

↳ if n is **even** use double-angle formulae

if n is **odd** split the power, use trig identify

C) $\int \sin^n(x)\cos^m(x)dx$ for both $m > 1$ and $n > 1$

↳ if one power is **odd**, split the odd power, use trig identity, reduce to A)

if both powers are **even**, use trig identify, reduce to B)

$$\int \sin^3(x)\cos^2(x)dx$$

$$\int \sin^5(x)dx$$

$$\int \sin^2(2x)\cos^2(x)dx$$

$\sin^n(x)\cos^m(x)$: what if powers are negative?

When we allow negative powers n and m, **logarithms** may appear.

$$\int \tan(x)\,dx$$

$\sin^n(x)\cos^m(x)$: what if powers are negative?

When we allow negative powers n and m, **logarithms** may appear.

$$\int \tan(x)\,dx = \int \frac{\sin x}{\cos x}\,\mathrm{dx}$$

$\sin^n(x)\cos^m(x)$: what if powers are negative?

When we allow negative powers n and m, **logarithms** may appear.

$$\int \tan(x)\,dx = \int \frac{\sin x}{\cos x}\,\mathrm{dx} \quad [\text{sub } u = \cos x]$$

$$= \int \frac{-du}{u} = -\ln|\cos x| = \ln|\sec x| + C$$

$\sin^n(x)\cos^m(x)$: what if powers are negative?

When we allow negative powers n and m, **logarithms** may appear.

$\int \tan(x)\,dx = \int \frac{\sin x}{\cos x}$ dx [sub u = cos x]

$= \int \frac{-du}{u} = -\ln|\cos x| = \ln|\sec x|$ +C

$\int \sec(x)$ dx

$\sin^n(x)\cos^m(x)$: what if powers are negative?

When we allow negative powers n and m, **logarithms** may appear.

$\int \tan(x)\,dx = \int \frac{\sin x}{\cos x}$ dx [sub u = cos x]

$= \int \frac{-du}{u} = -\ln|\cos x| + C = \ln|\sec x| + C$

$\int \sec(x)dx =$ [sub u = sec $x +$ tan x]

$= \ln|\sec x + \tan x| + C$

What about $\mathbf{sec}^{\boldsymbol{n}}(\boldsymbol{x})\mathbf{tan}^{\boldsymbol{m}}(\boldsymbol{x})$?

Basic antiderivatives

$\int \sec^2(x)dx$ = $\tan x + C$

$\int \tan(x)\sec(x)\,dx$ = $\sec(x)$ + C

Useful trig. Identity

$\tan^2(x)$ + 1 = $\sec^2(x)$

1) $\int \tan^m(x)\sec^2(x)dx$ for any m
2) $\int \tan^5(x)\sec^4(x)dx$
3) $\int \tan(x)\sec^n(x)\,dx$ for any n

1) For any m, $\int \tan^m(x)\sec^2(x)dx$ [sub u=tan(x)] = $\frac{1}{m+1}\tan^{m+1}(x) + C$

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2) $\int \tan^5(x)\sec^4(x)dx$

Keep $\sec^2(x)$, convert the other secant factor into powers of tangent

$=\int \tan^5(x)\sec^2(x)\sec^2(x)dx =$

$= \int \tan^5(x)(\tan^2(x) + 1)\sec^2(x)dx=$

$=\int \tan^7(x)\sec^2(x)dx + \int \tan^5(x)\sec^2(x)dx$ [sub u=tan(x)]

$=\int u^7\,du + \int u^5 du = \ldots$

1) For any m, $\int \tan^m(x)\sec^2(x)dx$ [sub u=tan(x)] = $\frac{1}{m+1}\tan^{m+1}(x) + C$

2) $\int \tan^5(x)\sec^4(x)dx$

Keep $\sec^2(x)$, convert the other secant factor into powers of tangent

$=\int \tan^5(x)\sec^2(x)\sec^2(x)dx =$

$= \int \tan^5(x)\,(\tan^2(x) + 1)\sec^2(x)dx=$

$=\int \tan^7(x)\sec^2(x)dx + \int \tan^5(x)\sec^2(x)dx$ [sub u=tan(x)]

$=\int u^7\, du + \int u^5 du = \ldots$

3) For any n,

$\int \tan(x)\sec^n(x)\,dx = \int \sec^{n-1}(x)\sec(x)\tan(x)\,dx$ [sub u =sec(x)]

$= \frac{1}{n}\sec^n(x) + C$

General case: $\int \tan^m(x)\sec^n(x)dx$

For any m, and $\boldsymbol{n}$ **even**

$\int \tan^m(x)\sec^n(x)dx = \int \tan^m(x)\sec^{n-2}(x)\sec^2(x)dx$

convert powers of secant into powers of tangent using

$\tan^2(x) + 1 = \sec^2(x)$ then sub $u = \tan(x)$

For any n, and $\boldsymbol{m}$ **odd**

$\int \tan^m(x)\sec^n(x)dx = \int \tan^{m-1}(x)\sec^{n-1}(x)\sec(x)\tan(x)\,dx =$

convert powers of tangent into powers of secant by using

$\tan^2(x) = \sec^2(x) - 1$

What if $\boldsymbol{n}$ **odd** and $\boldsymbol{m}$ **even?** Need new strategy…on Wednesday.

$$\int -zcos^{-2}(za)\sin(\tan(za))da =$$

A) $\cos(\tan(\mathrm{za})) + C$

B) $\mathrm{st}(\mathrm{eph}(\mathrm{en})) + C$

C) $\mathrm{j}(\mathrm{am}(\mathrm{es})) + C$

D) $\mathrm{rob}(\mathrm{er}(\mathrm{t})) + C$

E) None of the above