Integration by Parts

differentiation rule $\leftrightarrow$ integration technique
chain rule $\leftrightarrow$ substitution method
product rule $\leftrightarrow$ **integration by parts**

- product rule: $[u(x)v(x)]' = u(x)v'(x) + v(x)u'(x)$
- equivalently: $\int (u(x)v'(x) + v(x)u'(x))\, dx = u(x)v(x) + C$
- rearrange: $\boxed{\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx}$
  (where did C go?)
- compact notation: write $dv = v'(x)dx$, $du = u'(x)dx$:

$$\int u\, dv = u\, v - \int v\, du \qquad \text{Integration by Parts}$$

- may be useful when integrand is a product, and you can integrate one function: $dv \mapsto v$
  (and differentiate the other $u \mapsto du$)

$$\int u\,dv = uv - \int v\,du$$ Integration by Parts

Example: $\int x\ln(x)dx$

$u = \ln(x) \quad dv = x\,dx \quad \Longrightarrow \quad du = \frac{1}{x}dx \quad v = \frac{1}{2}x^2$

so: $\int x\ln(x)dx = \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C$

For definite integrals, the fundamental theorem of calculus implies

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]\big|_a^b - \int_a^b v(x)u'(x)dx$$

Example: $\int_0^{\frac{\pi}{2}} x\cos(x)dx \quad = x\sin(x)\big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin(x)dx$

$= \frac{\pi}{2} - 0 + \cos(x)\big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$

Try these:

1. $\int \ln(x) dx$

2. $\int_0^1 y^2 e^y dy$

3. $\int e^x \cos(x) dx$

4. $\int t^3 e^{-t^2} dt$

5. $\int_1^{\sqrt{3}} \arctan\left(\frac{1}{s}\right) ds$

The region $\{0 \le x \le 1,\ e^x \le y \le e\}$ is rotated about the y-axis. Find the resulting volume (by two different methods).

Shells (int. w.r.t. x): the height of the 'shell' at x is $e - e^x$, so

$$V = 2\pi \int_0^1 x(e - e^x)dx = 2\pi \left[e \int_0^1 xdx - \int_0^1 xe^x dx\right]$$
$$= 2\pi \left[e\frac{x^2}{2}|_0^1 - \left(xe^x|_0^1 - \int_0^1 e^x dx\right)\right]$$
$$= 2\pi \left[\frac{e}{2} - e + e^x|_0^1\right] = 2\pi \left[-\frac{e}{2} + e - 1\right] = \pi(e-2)$$

Disks (int. w.r.t. y): the disk at y has radius $x = \ln(y)$, so

$$V = \pi \int_1^e (\ln(y))^2 dy = \pi \left[y \ln^2(y)|_1^e - 2\int_1^e \ln(y)dy\right]$$
$$= \pi \left[e - 2\left(y\ln(y)|_1^e - \int_1^e dy\right)\right] = \pi\left[e - 2e + 2(e-1)\right]$$
$$= \pi(e-2)$$

Evaluate $\int_0^{\frac{\pi}{2}} \sin^2(x)dx$ by integration by parts.
Then (time permitting) evaluate $\int_0^{\frac{\pi}{2}} \sin^n(x)dx, \quad n \in \{1,2,3,\dots\}$.

$$\int_0^{\frac{\pi}{2}} \sin^2(x)dx = \int_0^{\frac{\pi}{2}} \sin(x)\ \sin(x)dx = -\sin(x)\cos(x)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos^2(x)d$$

$$= \int_0^{\frac{\pi}{2}} (1-\sin^2(x))dx = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin^2(x)dx$$

so $2\int_0^{\frac{\pi}{2}} \sin^2(x)dx = \frac{\pi}{2}$, and so $\int_0^{\frac{\pi}{2}} \sin^2(x)dx = \frac{\pi}{4}$.

Now set $I_n = \int_0^{\frac{\pi}{2}} \sin^n(x)dx = \int_0^{\frac{\pi}{2}} \sin^{n-1}(x)\sin(x)dx$:

$I_n = -\sin^{n-1}(x)\cos(x)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}(x)\cos^2(x)dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)(1-\sin^2(x))dx = (n-1)I_{n-2} - (n-1)I_n$,

so $nI_n = (n-1)I_{n-2}$, and so $\boxed{I_n = \frac{n-1}{n} I_{n-2}}$.

E.g.: $\int_0^{\frac{\pi}{2}} \sin^4(x)dx = I_4 = \frac{3}{4}I_2 = \frac{3}{4}\frac{\pi}{4} = \frac{3}{16}\pi$;

$\int_0^{\frac{\pi}{2}} \sin^3(x)dx = I_3 = \frac{2}{3}I_1 = \frac{2}{3}\int_0^{\frac{\pi}{2}} \sin(x)dx = \frac{2}{3}$.